UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 14 June, 2005 1.30 to 4.30

PAPER 72

GALAXIES AND DARK MATTER

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (1) We may study the vertical structure of a thin axisymmetric disk by neglecting all radial derivatives and adopting the form $f = f(E_z)$ for the distribution function, where $E_z \equiv \frac{1}{2}v_z^2 + \Phi(z)$. Show that if $f = \rho_0(2\pi\sigma_z^2)^{-1/2}\exp(-E_z/\sigma_z^2)$, the approximate form of Poisson's equation may be written:

$$2\frac{d^2\phi}{d\zeta^2} = e^{-\phi} \ , \text{ where } \phi \equiv \frac{\Phi}{\sigma_z^2} \ , \ \zeta \equiv \frac{z}{z_0} \text{ and } z_0 \equiv \frac{\sigma_z}{\sqrt{8\pi G\rho_0}} \ .$$

By solving this equation subject to the boundary conditions

$$\phi(0) = d\phi/d\zeta|_0 = 0 \ ,$$

show that the density ρ in the disk is given by

$$\rho(z) = \rho_0 \text{sech}^2\left(\frac{z}{2z_0}\right) \ .$$

(2) Consider a spherically-symmetric stellar-dynamical system with distribution function

$$f(E) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}}\exp[E/\sigma^2] \ ,$$

where $E = \Psi - \frac{1}{2}v^2$ is the relative energy, where Ψ is the potential, v is velocity, σ is the (constant) dispersion, and ρ_1 is a constant. Find the density ρ of the system, and write down the Poisson equation for the system.

The equation of hydrostatic support for an isothermal gas of density $\rho(r)$ at temperature T is

$$\frac{kT}{m_0}\frac{d\rho}{dr} = -\rho\frac{Gm(r)}{r^2} \ ,$$

where k is Boltzmann's constant, m_0 is the mass per particle, and $m(r)$ is the total mass interior to radius r. Show that the stellar dynamical system and the gaseous system have the same density structure when

$$\sigma^2 = \frac{kT}{m_0} \ .$$

Note that: $\int_0^\infty \exp(-\alpha x^2)dx = \sqrt{\pi/\alpha}$.

2 Imagine a particle of mass M moving through an equilibrium, spherically symmetric system of particles of mass $m, M > m$. The distribution function of the particles of mass m is $f(v) = \frac{n_0}{(2\pi\sigma^2)^{3/2}} \left\{\exp(-v^2/2\sigma^2) - \exp(-v_e^2/2\sigma^2)\right\}$ with v a particle velocity, v_e the escape velocity, V an rms velocity, σ the isotropic velocity dispersion, $n_0 = \rho_0/m$ a number density, and ρ_0 the density within a core

$$\rho = \rho_0 \ , \quad \text{for } r < r_0 = (9\sigma^2/4\pi G\rho)^{1/2} \ .$$

a) Show that the massive particle is systematically slowed by dynamical friction at a rate

$$\frac{dV}{dt} = -16\pi^2 G^2 \frac{mM}{V^2} \ln\Lambda \int_0^V f(v) v^2 dv$$

where b_m is the maximum impact parameter relevant to the situation, and $\Lambda = \frac{b_m V^2}{GM}$.

b) Show the escape velocity and circular velocity are related by

$$v_e^2(r) = V_c^2(r)\left(\frac{3r_0^2}{r^2} - 1\right)$$

c) Assume M is in a circular orbit, so that $v \equiv V \equiv V_c$. With the substitution $X = v_c/\sqrt{2}\sigma$, and defining τ, the characteristic time scale with which the orbit decays, by

$$\tau = \frac{r}{2(dr/dt)_r}$$

show that

$$\tau = \frac{V_c^3}{32\pi^2 \ln\Lambda G^2 M \rho_0 I(X)}$$

where $I(X) = \frac{1}{4\pi}\left\{\text{erf}(X) - \frac{2X}{\sqrt{\pi}}\exp(-X^2) - \frac{4X^3}{3\sqrt{\pi}}\exp(X^2 - 9/2)\right\}$,
and $\text{erf}(X) \equiv \frac{2}{\sqrt{\pi}}\int_0^X e^{-t^2} dt$.

3 The hydrostatic equilibrium Jeans' equation in a spherical galaxy, with coordinates (r, θ, ϕ), density ρ, and potential Φ and mass $M(r)$, is

$$\frac{\partial\left(\rho\overline{v_r^2}\right)}{\partial r} + \frac{\rho}{r}\left\{2\overline{v_r^2} - (\overline{v_\theta^2} + \overline{v_\phi^2})\right\} = -\rho\frac{\partial\Phi}{\partial r} .$$

a) Consider a spherical isotropic system, with density $\rho(r) = \rho_0\left(\frac{r_0}{r}\right)^k$, r_0 a scale factor. Determine the velocity dispersion in the two cases (i) $k < 1$; and (ii) $k > 1$. You may find it helpful to consider the dominant range of contributions to the integral.

b) Consider a spherical system with anisotropic velocity dispersion defined by $\beta = 1 - \overline{v_\theta^2}/\overline{v_r^2}$, observed at projected distance R from its centre. $I(R)$ is the projected surface brightness distribution, σ_p^2 the projected velocity dispersion. Show that

$$I\sigma_p^2 - R^2\int_R^\infty \frac{\rho GM(r)dr}{r^2\sqrt{r^2 - R^2}} = \int_R^\infty \left\{2\rho\overline{v_r^2} + \frac{R^2}{r}\frac{\partial(\rho\overline{v_r^2})}{\partial r}\right\}\frac{rdr}{\sqrt{r^2 - R^2}} .$$

Discuss the implications for deducing the existence of central massive black holes in galaxies from surface brightness and velocity dispersion profiles.

4 (a) Define the following properties of an orbit

(i) constant of the motion

(ii) integral of the motion

(iii) isolating integral of the motion

Under which conditions are there the following numbers of isolating integrals for an orbit

(iv) zero

(v) one

(vi) more than three

(vii) five.

(b) Consider the general spherical gravitational potential

$$\Phi(r) = -GM\left(\frac{1}{r} + \frac{a}{r^2}\right) ,$$

where M is a mass and a a constant. In this potential the equations of motion may be written as

$$\frac{d^2u}{d\psi^2} + \left(1 - \frac{2GMa}{L^2}\right)u = \frac{GM}{L^2}$$

where u is the inverse radial coordinate, ψ an angular coordinate, and L the orbit's angular momentum.

Consider this to determine under which conditions five isolating integrals of the motion exist.

END OF PAPER