UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 12

PAPER 70

STRUCTURE AND EVOLUTION OF STARS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

The notation used is standard and you are reminded of the equations of stellar structure in the form:

$$\frac{dP}{dr} = -\frac{\mathrm{G}m\rho}{r^2};$$

$$\frac{dm}{dr} = 4\pi r^2\rho;$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acr^2T^3};$$

$$\frac{dL_r}{dr} = 4\pi r^2\rho\epsilon;$$

$$P = \frac{\Re\rho T}{\mu} + \frac{1}{3}aT^4.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 By considering the adiabatic displacement of a packet of fluid, derive the Schwarzschild criterion for convective stability and show that, for an ideal gas with ratio of specific heats $\gamma = 5/3$, the fluid is unstable to convection if

$$\nabla = \frac{d \log T}{d \log P} > \frac{2}{5},$$

where P is the pressure and T is the temperature.

In a thin stellar atmosphere

$$T^4 = \frac{3}{4} T_{\mathrm{e}}^4 \left(\tau + \frac{2}{3} \right),$$

where T_{e} is the effective temperature and

$$\tau = \int_r^{\infty} \kappa \rho \, dr$$

is the optical depth and ρ is the density. The atmosphere of a cool star behaves approximately as an ideal gas with $\gamma = 5/3$ and has opacity

$$\kappa = \kappa_0 P^{1/2} T^8,$$

where κ_0 is a constant. Explain briefly what conditions lead to this dependence.

Show that

$$P^{3/2} = \frac{8GM}{\kappa_0 R^2 T_{\mathrm{e}}^8} \left\{ \frac{1}{2} - \frac{1}{3\tau + 2} \right\},$$

where M is the mass of the star and R is its radius.

Find the value of $\tau = \tau_{\mathrm{c}}$ at which convection sets in.

The star is fully convective for $\tau > \tau_{\mathrm{c}}$ so that the pressure and density are related by $P = K\rho^{5/3}$ with K a constant for each star. Deduce that

$$L \propto R^{98/47} M^{28/47}.$$

[For an $n = 3/2$ polytrope you may assume that $R \propto KM^{-1/3}$]

2 Estimate the mean kinetic energy $\langle E \rangle$ for a proton in the centre of the Sun and compare it with the Coulomb energy $E_{\rm C}$ owing to the electrostatic repulsion that must be overcome in bringing two protons together.

State briefly the two physical ideas that allow this barrier to be surmounted.

Show that in a collision between two protons, each of mass $m_{\rm p}$, the kinetic energy E in the centre of mass frame is related to the relative velocity v by $E = \frac{1}{4} m_{\rm p} v^2$.

The cross-section for nuclear reactions between two protons can be written in the form

$$\sigma(E) = \frac{S_0}{E} \exp\left\{ -2\sqrt{\frac{E_{\rm B}}{E}} \right\},$$

where S_0 is a constant and $E_{\rm B}$ is the quantum mechanical barrier energy. Explain very briefly how the terms in this expression arise.

For non-degenerate, non-relativistic gas at temperature T the relative velocity distribution is Maxwellian given by

$$n(v)\, dv = 4\pi \left(\frac{m_{\rm p}}{4\pi kT} \right)^{\frac{3}{2}} \exp\left(-\frac{E}{kT} \right) v^2\, dv.$$

The number density of reacting particles is N. Show that the reaction rate $R_{\rm pp}$ per unit volume per unit time is

$$R_{\rm pp} = \frac{1}{2} N^2 \int_0^\infty v\sigma(v) n(v)\, dv.$$

Deduce that

$$R_{\rm pp} = \frac{S_0 N^2}{(kT)^{3/2}} \left(\frac{4}{\pi m_{\rm p}} \right)^{\frac{1}{2}} \int_0^\infty \exp\left\{ -\frac{E}{kT} - 2\sqrt{\frac{E_B}{E}} \right\} dE.$$

Find the Gamow energy $E_{\rm G}$ at which the integrand peaks and show that $kT \ll E_{\rm G} \ll E_{\rm B}$.

Approximate the integrand by a Gaussian centred on $E_{\rm G}$ and deduce that the temperature dependence of the reaction rate takes approximately the form

$$R_{\rm pp} \propto \frac{1}{T^\alpha} \exp\left\{ -\left(\beta/T\right)^{\frac{1}{3}} \right\},$$

where α and β are constants which you should determine.

[The temperature at the centre of the Sun $T_{\rm c} = 2 \times 10^7$ K, $k = 1.4 \times 10^{-16}\,{\rm erg\,K^{-1}}$, the electrostatic force between two protons is e^2/r^2 where $e^2 = 2.3 \times 10^{-19}$ in cgs units and the radius of a proton $r_{\rm p} = 10^{-13}$ cm.]

3 Show that in a frame in which all the material is corotating with angular velocity $\boldsymbol{\Omega}$ the equation of hydrostatic equilibrium of a star can be written as

$$\nabla P = -\rho \nabla \phi$$

for pressure P, density ρ and combined gravitational and centrifugal potential $\phi(\boldsymbol{r})$ which satisfies

$$\nabla^2 \phi = 4\pi G \rho - 2\Omega^2 .$$

Show that P and ρ must be constant on equipotential surfaces. Hence deduce that $\nabla^2\phi$, but not necessarily $|\nabla\phi|$, are constant on equipotential surfaces.

Argue that, for a star of uniform composition, temperature T is also constant on equipotential surfaces.

The star is in radiative equilibrium with heat flux

$$\boldsymbol{F} = -\chi \nabla T = -\chi \frac{dT}{d\phi} \nabla \phi,$$

where χ is the conductivity which is related to the opacity $\kappa(\rho, T)$ by

$$\chi = \frac{4acT^3}{3\kappa\rho},$$

a is the radiation constant and c is the speed of light. Show that the effective temperature on the surface of the star

$$T_{\mathrm{e}} \propto g^{1/4},$$

where g is the magnitude of the effective gravitational acceleration and sketch the cross-section of a rapidly rotating star and indicate where it is hottest.

Why is it not in general possible for the energy balance to be given simply by

$$\nabla . \boldsymbol{F} = \rho \epsilon,$$

where $\epsilon(\rho, T)$ is the energy generation rate per unit mass?

Now suppose that there is a steady circulation velocity field $\boldsymbol{v}(\boldsymbol{r})$ so that the energy balance is given instead by

$$\rho T \frac{Ds}{Dt} = \rho \boldsymbol{v} . T \nabla s = \rho \epsilon - \nabla . \boldsymbol{F},$$

where $s(\rho, T)$ is the specific entropy. Use continuity and the thermodynamic relation

$$T\, ds = dh - \frac{1}{\rho}\, dP,$$

where $h(\rho, T)$ is the specific enthalpy, to show that

$$\int_S \boldsymbol{F} . \boldsymbol{dS} = \int_V \rho \epsilon \, dV,$$

where S is an equipotential surface enclosing volume V.

Hence show that the radiative gradient is given by

$$\frac{d \log T}{d \log P} = \frac{3\kappa P L}{16\pi a c G m T^4} \left(1 - \frac{\Omega^2 V}{2\pi G m}\right)^{-1},$$

where L is the rate of energy generation within V and m is the mass in V.

4 A red giant of mass M_1 is in a binary system with a main-sequence star of mass M_2. The red giant is losing mass in a fast spherically symmetric stellar wind at a rate $\dot{M} < 0$. Show that, if the intrinsic angular momentum of the stars is neglected,

$$\frac{\dot{J}_{\text{orb}}}{J_{\text{orb}}} = \frac{M_2\dot{M}}{M_1 M},$$

where $M = M_1 + M_2$ and that the orbital period P and separation a obey

$$P \propto M^{-2}, \quad a \propto M^{-1}$$

On a short timescale the radius of the giant R_1 responds according to

$$R_1 \propto M_1^{-n} \quad 0 < n < 1$$

and the radius of its Roche lobe is approximated by

$$\frac{R_{\text{L}}}{a} = 0.426 \left(\frac{M_1}{M}\right)^{\frac{1}{3}}.$$

Now suppose that the giant is filling its Roche lobe and that wind mass loss is taking place on a timescale much shorter than the nuclear timescale. Show, by differentiating $\log(R_1/R_{\text{L}})$ or otherwise, that mass transfer is driven by the wind if

$$q = \frac{M_1}{M_2} < \frac{1+3n}{3(1-n)} \qquad (\dagger).$$

What happens otherwise?

Show further that, when $(\dagger)$ is satisfied and $6q < 5 - 3n$, the rate of mass transfer to the main-sequence star

$$\dot{M}_2 = -\frac{1 + 3n - 3(1-n)q}{(1+q)(5-3n-6q)}\dot{M}.$$

What is the physical consequence if $6q > 5 - 3n$?

END OF PAPER