UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 9 to 11

PAPER 67

FOURIER TRANSFORMS, THEIR GENERALISATIONS AND THE IMAGING OF THE BRAIN

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $q(x,t)$ satisfy the following initial-boundary value problem

$$q_t + q_{xxx} = 0, \quad 0 < x < L, t > 0, \tag{1}$$
$$q(x,0) = q_0(x), \quad 0 < x < L, \tag{2}$$
$$q(0,t) = f_0(t), \quad q(L,t) = g_0(t), \quad q_x(L,t) = g_1(t), t > 0, \tag{3}$$

where the functions $q_0(x), f_0(t), g_0(t), g_1(t)$, are sufficiently smooth and are compatible at $x = t = 0$ and at $x = L, t = 0$.

(a) Write equation (1) in the form

$$(e^{-ikx+\omega(k)t}q)_t - \left(e^{-ikx+\omega(k)t}X\right)_x = 0,\ k \in C,$$

where $\omega(k)$ and X are to be determined.

(b) Write the global relation and the equations obtained from the global relation via certain invariant transformations.

(c) Find an integral representation for $q(x,t)$ in the complex k-plane, involving appropriate x- and t- transforms of all boundary values.

(d) Discuss, without giving any details, of how to use the equations obtained in (b) to prove that the IBV problem (1)-(3) is well-posed.

(e) By computing the determinant of the system of (b) and by analysing its zeros, show that there is no discrete spectrum for the IBV problem (1)-(3).

2 Let the scalar function $\mu(x_1, x_2, \lambda)$ satisfy

$$\frac{1}{2}(\lambda + \frac{1}{\lambda})\frac{\partial \mu}{\partial x_1} + \frac{1}{2i}(\lambda - \frac{1}{\lambda})\frac{\partial \mu}{\partial x_2} = f(x_1, x_2),\ -\infty < x_1, x_2 < \infty, \lambda \in C, \tag{1}$$

where $f(x_1, x_2)$ has sufficient smoothness and also decays as $|x_1|^2 + |x_2|^2 \to \infty$.

(a) Use an appropriate change of variables to reduce equation (1) to

$$\nu(|\lambda|)\frac{\partial \mu}{\partial \bar{z}} = f(x_1, x_2),\ |\lambda| \neq 1, \tag{2}$$

where $\nu(|\lambda|)$ and z are to be determined.

(b) By supplementing equation (2) with the boundary condition

$$\mu = O\left(\frac{1}{z}\right),\ z \to \infty, \tag{3}$$

find the unique solution of equations (2) and (3).

(c) Use the solution obtained in (b) to show that

$$\mu(x_1, x_2, \lambda^+) = -P^- \widehat{f}(\rho, \theta) - \int_\tau^\infty F(\tau', \rho, \theta) d\tau', \tag{4}$$

where $\lambda^- = \lim_{\varepsilon \to 0}(1 + \varepsilon)e^{i\theta},\ 0 \leqslant \theta \leqslant 2\pi,\ \varepsilon > 0,\ P^-$ denotes the usual projector in the variable $\rho, \widehat{f}$ denotes the Radon transform of f, and F denotes f written in local coordinates.

(d) By assuming that $\mu(x_1, x_2, \lambda^-)$ satisfies an equation similar to (4) but with P^- replaced with $(-P^+)$, obtain the inverse Radon transform.

(e) Discuss the usefulness of the above result in computed tomography.

(f) Discuss, without giving any details, the usefulness of the above result in Single Photon Emission Computerised Tomography.

3 Let D denote the interior of the equilateral triangle with corners at the points

$$z_1 = \frac{le^{\frac{i\pi}{3}}}{\sqrt{3}}, \quad z_2 = \bar{z}_1, \quad z_3 = -\frac{l}{\sqrt{3}},$$

where z denotes the usual complex variable $z = x + iy$ and l is a positive constant. The sides $(z_2, z_1), (z_3, z_2), (z_1, z_3)$, will be referred to as (1), (2), (3) respectively. The functions $q^{(j)}(s)$ and $q_N^{(j)}(s)$ denote the Dirichlet and the Neumann boundary values of side (j).

Let the real-valued function $q(x, y)$ satisfy the PDE

$$q_{xx} + q_{yy} - 4\lambda q = 0, \quad (x, y) \in D, \ \lambda \in R, \quad \lambda \text{ real constant}, \tag{1}$$

with Neumann boundary values

$$q_N^{(j)}(s) = f(s), \quad s \in [-\frac{l}{2}, \frac{l}{2}], \quad j = 1, 2, 3. \tag{2}$$

(a) Show that equation (1) can be written in the form

$$\left(e^{-ikz - \frac{\lambda}{ik}\overline{z}} A(z, \overline{z})\right)_{\overline{z}} - \left(e^{-ikz - \frac{\lambda}{ik}\overline{z}} B(z, \overline{z}, k)\right)_z = 0, \tag{3}$$

where $A(z, \bar{z})$ and $B(z, \bar{z}, k)$ are to be determined.

(b) Show that the proper spectral functions are given by

$$\rho_j(k) = E(-ik)[\frac{i}{2}\Psi_j(k) + \Phi_j(k)], \quad j = 1, 2, 3, \tag{4}$$

where $E(k), \Psi_j(k), \Phi_j(k)$ are to be determined.

(c) Write the global relation and show that for the boundary conditions (2) it becomes one equation with three unknowns.

(d) Solve the global relation and its Schwartz conjugate for $\Phi(\overline{a}k)$, where $a = e^{\frac{2i\pi}{3}}$.

(e) By evaluating the expression obtained in (d) at appropriate values of k, compute $q(s)$.

The following elementary identities are useful

$$1 + a + \overline{a} = 0, \quad i\overline{a} - ia = \sqrt{3}, \quad ia - i = \sqrt{3}\overline{a}.$$

END OF PAPER