UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 6 June, 2005 9 to 12

PAPER 65

BIFURCATIONS AND INSTABILITIES IN DISSIPATIVE SYSTEMS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Newell–Whitehead–Segel equation

$$A_T = \mu A + \left(\partial_X + \frac{\mathrm{i}}{2}\partial^2_{YY}\right)^2 A - A|A|^2,$$

describes the evolution on long time and space scales of the amplitude $A(X, Y, T)$ of a steady weakly nonlinear pattern in an isotropic 2D medium.

(a) Show that uniform amplitude patterns $A(X, T) = R_0 \mathrm{e}^{\mathrm{i}qX}$ exist for $q^2 \leqslant \mu$. Using the Lyapounov functional

$$V = \left\langle \mu|A|^2 - \frac{1}{2}|A|^4 - \left|\left(\partial_X + \frac{\mathrm{i}}{2}\partial^2_{YY}\right) A\right|^2 \right\rangle,$$

where $\langle f \rangle \equiv \frac{1}{L}\int_0^L f\mathrm{d}x$, discuss the possibility of a 'zig-zag' instability of uniform amplitude patterns with $q < 0$. Comment on the case $q \geqslant 0$.

(b) By symmetry arguments justify the form of the phase equation

$$\phi_T = \lambda\phi_{YY} - \gamma\phi_{YYYY} - \alpha\left(\phi_Y\right)^2\phi_{YY}, \qquad (*)$$

for the nonlinear evolution of the zig-zag instability.

(c) Let $w_0(x)$ be a steady (but not necessarily small amplitude) solution of the Swift–Hohenberg equation

$$w_t = rw - (1 + \nabla^2)^2 w - w^3,$$

where $\nabla^2 \equiv \partial^2_{xx} + \partial^2_{yy}$. By looking for solutions modulated in the y direction on a long length scale $Y = \varepsilon y$, i.e.

$$w = w_0(x + \phi(Y, T)) + \varepsilon^2 w_2(x, Y, T) + \varepsilon^4 w_4(x, Y, T) + \cdots,$$

develop a multiple-scales expansion in powers of ε. Deduce the equation

$$\mathcal{L}w_2 = 2[(w_0'''' + w_0'')(\phi_Y)^2 + (w_0''' + w_0')\phi_{YY}],$$

where $\mathcal{L}$ is the linear operator

$$\mathcal{L}u \equiv ru - (1 + \partial^2_{xx})^2 u - 3w_0^2 u.$$

Give the relevant solvability condition. Show that the solvability condition at $O(\varepsilon^2)$ is satisfied if

$$\left\langle (w_0')^2 + w_0' w_0''' \right\rangle = O(\varepsilon^2).$$

(d) Let the bifurcation parameter λ be defined by

$$-\varepsilon^2\frac{\lambda}{2}\left\langle (w_0')^2 \right\rangle = \left\langle (w_0')^2 + w_0' w_0''' \right\rangle,$$

and assume that the solution for w_2 may be written in the form

$$w_2(x, Y, T) = C_1(x)\phi_{YY} + C_2(x)(\phi_Y)^2,$$

(you are not expected to solve for $C_1(x)$ and $C_2(x)$ explicitly). Show that the evolution equation for $\phi(Y, T)$ is the phase equation $(*)$ with α and γ taking the values

$$\alpha = \frac{6\,\langle w_0'(w_0''' + w_0 C_1 C_2)\rangle}{\langle (w_0')^2\rangle},$$
$$\gamma = \frac{\langle w_0'(2C_1 + 2C_1'' + w_0')\rangle}{\langle (w_0')^2\rangle}.$$

2 Researcher A uses the differential equations

$$\dot{x} = \left(\frac{1}{a} - 1\right) x - \frac{1}{a}x^2 - xy,$$
$$\dot{y} = \frac{1}{b}xy - y,$$

for $x \geqslant 0$ and $y \geqslant 0$ to describe a particular predator–prey interaction, with two real parameters $a > 0$ and $0 < b \leqslant 1$.

(a) Find the equilibrium points and investigate their stability. Hence sketch phase portraits of the quadrant $x \geqslant 0$, $y \geqslant 0$ in each of the three regions of the (a, b) plane that show qualitatively different dynamics.

(b) Let A be a 2×2 real matrix with trace T and determinant D. Show that A has an eigenvalue of 1 if $T = D + 1$. Show also that A has an eigenvalue of -1 if $T + D + 1 = 0$, and that A has complex eigenvalues $\mathrm{e}^{\pm \mathrm{i}\alpha}$ and $\alpha \neq 0, \pi$ if $D = 1$ and $|T| < 2$.

(c) Researcher B decides that a better model of the dynamics is given by the 2D map

$$x_{n+1} = \frac{1}{a}x_n(1 - x_n) - x_n y_n,$$
$$y_{n+1} = \frac{1}{b}x_n y_n,$$

where, again, $a > 0$ and $0 < b \leqslant 1$. Locate the fixed points of the map and investigate bifurcations from these fixed points, using the results of part (b) if you wish.

Divide the (a, b) parameter space into regions of qualitatively different dynamics and compare this with your answer to part (a).

(d) Briefly describe the dynamics of the 2D map close to the line $a + 2b = 1$ for $1/9 < a < 1$.

3 A 1D pattern forming system undergoes an oscillatory instability with frequency ω_0 at finite wavenumber k_c, from an initially time independent, reflection-symmetric and spatially uniform state.

(a) Show, using symmetry arguments, that the small-amplitude weakly nonlinear behaviour near the instability is governed by amplitude equations of the form

$$\begin{aligned} A_T &= \mu A - \beta A|A|^2 - \gamma A|B|^2, \\ B_T &= \mu B - \beta B|B|^2 - \gamma B|A|^2, \end{aligned} \qquad (*)$$

where $\beta \equiv \beta_r + \mathrm{i}\beta_i \in \mathbb{C}$, $\gamma \equiv \gamma_r + \mathrm{i}\gamma_i \in \mathbb{C}$, and $\mu \in \mathbb{R}$ and variations of $A(T)$ and $B(T)$ on long length scales have been ignored.

Show that solutions in the form of travelling and standing waves exist and determine the conditions on β_r and γ_r for their linear stability.

(b) The pattern forming system is now subjected to time-periodic forcing with frequency $\omega_f = 2\omega_0$. Show that this breaking of the continuous time translation symmetry allows the occurrence of extra terms $\nu\bar{B}$ and $\nu\bar{A}$ in the two original amplitude equations $(*)$, respectively, leading to the amplitude equations:

$$\begin{aligned} A_T &= \mu A + \nu\bar{B} - \beta A|A|^2 - \gamma A|B|^2, \\ B_T &= \mu B + \nu\bar{A} - \beta B|B|^2 - \gamma B|A|^2. \end{aligned} \qquad (**)$$

(c) Assume ν is real. By writing $A = R\mathrm{e}^{\mathrm{i}\theta}$ and $B = S\mathrm{e}^{\mathrm{i}\phi}$, show that equilibrium solutions of $(**)$ exist when

$$\begin{aligned} (\beta_i + \gamma_i)R^2 + \nu\sin(\theta+\phi) &= 0, \\ \mu - (\beta_r + \gamma_r)R^2 + \nu\cos(\theta+\phi) &= 0. \end{aligned}$$

Hence show that solutions exist when

$$\left|\frac{\mu\cos\chi}{\nu}\right| \quad \leqslant \quad 1,$$

where $\tan\chi = (\beta_r + \gamma_r)/(\beta_i + \gamma_i)$.

Briefly interpret the effect of sufficiently strong external forcing on a system undergoing an oscillatory instability of this kind.

4 A second order system of ODEs near a codimension-2 bifurcation is described by the following normal form:

$$\dot{x} = y,$$
$$\dot{y} = -\lambda + \mu y + x^2 + xy,$$

where λ and μ are real parameters.

(a) Find the equilibria and discuss their local bifurcations. Sketch the locations of these local bifurcations in the (λ, μ) plane. Discuss the existence of global bifurcations.

(b) Adopt the scaling $x = \varepsilon^2 u$, $y = \varepsilon^3 v$, $\lambda = \varepsilon^4 \alpha$, $\mu = \varepsilon^2 \beta$ and rescale time by a factor of $1/\varepsilon$. Find the resulting ODEs for u and v. In the limit $\varepsilon \to 0$, show that the quantity $H(u, v) = \frac{1}{2}v^2 + \alpha u - \frac{1}{3}u^3$ is conserved. Sketch contours of constant H in the (u, v) plane and find the value of H that corresponds to a homoclinic orbit in the system.

(c) Show that for $\varepsilon = 0$ an explicit solution for the homoclinic orbit is

$$u(t) = 1 - 3\,\mathrm{sech}^2\left(\frac{t}{\sqrt{2}}\right),$$
$$v(t) = 3\sqrt{2}\,\mathrm{sech}^2\left(\frac{t}{\sqrt{2}}\right)\tanh\left(\frac{t}{\sqrt{2}}\right).$$

(d) By integrating around the homoclinic orbit for small ε find the relation between α and β and hence between λ and μ at the global bifurcation, when ε is small. Show this line of global bifurcations in your sketch of the (λ, μ) plane and, assuming that there are no further bifurcations in the ODEs, sketch phase portraits in each region of the (λ, μ) plane that contains qualitatively different dynamics.

You may find the following result useful: $\int_{-\infty}^{\infty} \mathrm{sech}^2\tau \ \tanh^k \tau \ \mathrm{d}\tau = 2/(k+1)$.

END OF PAPER