UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 1.30 to 4.30

PAPER 63

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt **FOUR** questions.*

*There are **SEVEN** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a fibre bundle and a principal fibre bundle. What is a section of a bundle? Show that a principal bundle is trivial if and only if it admits a global section.

What is the frame bundle of a manifold? What is a pseudo-orthonormal frame bundle? Give a group-theoretic description of the orthonormal frame bundles of Minkowski spacetime and of De-Sitter spacetime.

Show that the frame bundle of any connected Lie Group is trivial. What can you say about the tangent bundle of a Lie group?

2 In five-dimensional supergravity, whose bosonic fields comprise the spacetime metric and a vector field, one adds to the usual action functional for an exact two-form $F = dA$,

$$-\frac{1}{2}\int F \wedge \star F,$$

a so-called Chern-Simons term of the form

$$c\int A \wedge F \wedge F\,,$$

where c is a constant.

Obtain the field equation for F. Explain why the field equation is gauge-invariant despite the the fact that the Chern-Simons integrand is not invariant under gauge transformations. The energy momentum tensor $T_{\mu\nu}$ for any field theory in a spacetime with metric $g_{\mu\nu}$ is given by the functional derivative

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\,.$$

Obtain the contribution to the energy momentum tensor from the two-form action, including any which arises from the Chern-Simons term.

3 A three-dimensional Lie group G is defined by the multiplication law

$$(x',y',z')(x,y,z) \to (x' + e^{-z'}x, y' + e^{z'}y, z + z') .$$

By considering (x', y', z') to be infinitesimal, verify that that the generators of left actions may be taken to be

$$R_1 = \partial_x , \qquad R_2 = \partial_y , \qquad R_3 = \partial_z - x\partial_x + y\partial_y .$$

Show also that the generators of right actions may be taken to be

$$L_1 = e^{-z}\partial_x , \qquad L_2 = e^{z}\partial_y , \qquad L_3 = \partial_z .$$

What are the Lie brackets

$$[L_i, R_j] \, ?$$

Why is this so?

By constructing a basis of one-forms dual to L_1, L_2, L_3, show that the metric on G given by

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2 ,$$

is a left-invariant metric.

4 Obtain the Jacobi identity for a Lie Algebra $\mathfrak{g}$, with structure constants $C_a{}^b{}_c$, in the form

$$C_{[a}{}^{e}{}_{|c|}C_{b}{}^{|c|}{}_{d]} = 0 ,$$

where $|c|$ indicates that the index c is omitted from the anti-symmetrization.

Assuming that the Lie algebra $\mathfrak{g}$ is *three-dimensional*, show that one may replace $C_a{}^b{}_c$ by a symmetric contravariant tensor $n^{ef} = n^{fe} \in \mathfrak{g}^\star \otimes_{\rm Sym} \mathfrak{g}^\star$, and a one-form $a_f \in \mathfrak{g}^\star$ such that

$$C_a{}^b{}_c \epsilon^{acd} = n^{bd} + \epsilon^{bde} a_e ,$$

where $\epsilon^{abc} = \epsilon^{[abc]}$ is a volume form on $g^\star$. Hence show that the Jacobi identity will be satisfied if and only if

$$n^{ae}a_e = 0 .$$

5 Define a symplectic manifold and a Poisson manifold. Give examples, in particular, examples of manifolds which are Poisson but not symplectic.

By showing that every symplectic manifold has an everywhere non-vanishing volume form, prove that every symplectic manifold is orientable.

Show that the cotangent bundle $T^\star(M)$ of a manifold M is a symplectic manifold.

Give a condition on a Poisson manifold that the Jacobi identity holds for the Poisson bracket defined on functions. Show that this condition is satisfied on a symplectic manifold.

6 A group G acts on a symplectic manifold $\{P, \omega\}$ preserving the symplectic form ω. Show how one obtains a moment map $\mu : P \to \mathfrak{g}^\star$. Give a necessary condition that the Poisson algebra of the moment maps coincides with the Lie algebra $\mathfrak{g}$ of the group G. Show that this condition is satisfied if the Killing form, or Killing metric, of $\mathfrak{g}$ is non-degenerate. For what groups is this latter property true?

Illustrate your answer by reference to the isotropic simple harmonic oscillator in three spatial dimensions with Hamiltonian

$$H = \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\mathbf{x}^2 \,.$$

Show in particular that there is such an action of $U(3)$ on $T^\star(\mathbb{R}^3)$, and that the moment map for the $U(1)$ factor is the Hamiltonian. Exhibit some other moment maps. Which ones arise from the obvious geometric action of $SO(3)$ on $\mathbb{R}^3$? Which ones do not?

7 Write an essay, either on *one* of the following topics

Geometric Quantization, in which case, your essay should give the aims and motivations behind the idea, describe the prequantization construction and discuss the problem of finding a suitable polarization.

or

Applications of Stokes's Theorem, in which case you should indicate briefly how Stokes's theorem works and show how it is used to construct gauge-invariant equations of motion for p-branes and how, using the Brouwer degree construction, it may be used to obtain topological conservation laws, giving as an example the Skyrmion, i.e. a theory based on an $SU(2)$ target space.

END OF PAPER