UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 1.30 to 3.30

## PAPER 61

## COSMOLOGY

*Attempt* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** In the early Universe the Friedmann equation, with zero cosmological constant is

$$H^2 = \frac{8\pi}{3}\, G\rho - \frac{k}{a^2}\,,$$

and the Raychaudhuri equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\, G(\rho + 3P)\,.$$

Consider a closed universe ($k > 0$) dominated by pressureless matter ($P = 0$). By considering conformal time, $\tau$, show that

$$a(\tau) = \frac{\Omega_0}{2(\Omega_0 - 1)}\,[1 - \cos(\sqrt{k}\tau)]$$
$$t(\tau) = H_0^{-1}\,\frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}\,[\sqrt{k}\tau - \sin(\sqrt{k}\tau)]$$

where $\Omega_0$ is the present density parameter, $H_0$ the present Hubble parameter and the normalisation is such that the scalefactor today is unity ($a(\tau_0) = 1$).

Hence obtain an expression for the present age, $t_0$, of the Universe for this model. Determine the time, $t_{BC}$, when the Universe ends in a big crunch.

**2** In the early Universe the Hubble constant is

$$H = 1.66\ g^{*1/2}\ T^2_{/M_{Pl}}\ ,$$

where the Planck mass, $M_{Pl} \sim 10^{19}$GeV. For temperatures $T > 1$MeV neutrons and protons are in thermal equilibrium via the interactions $\nu + n \leftrightarrow p + e^+$, with an interaction rate $\Gamma \sim G_F^2 T^5$, where $G_F$ is the Fermi constant given by $G_F \approx 10^{-5} m_p^{-2}$ and the proton mass, $m_p$ is $\sim 1$GeV. If the effective number of degrees of freedom at this time is $g^* = 10.75$, estimate the decoupling temperature, $T_d$, of neutrinos. Explain why the neutron to proton ratio at neutrino decoupling is given by

$$\frac{n_n}{n_p} = \frac{X_n}{X_p} = \exp(-Q_{/T_d})$$

where $Q = m_n - m_p$ and $X_i \equiv \dfrac{n_i}{n_B}$, where $n_B$ is the baryon number of the Universe.

Subsequently deuterium forms through the interaction $n + p \leftrightarrow D + \gamma$. Estimate the ratio of the fractional deuterium abundance, $X_D/X_n X_p$, in terms of the binding energy, $B = m_D - m_n - m_p$, and the baryon to photon ratio $\eta = \dfrac{n_B}{n_\gamma}$.

Roughly estimate the $^4$He abundance. Why is $^4$He synthesised at about $T \sim 0.1$MeV? If the neutron proton mass difference were increased by 10% how would this affect the $^4$He abundance?

**3** In an FRW universe a real scalar field $\phi$ has potential $V(\phi) = \frac{\lambda}{4}\phi^4$, where $\lambda$ is a small positive constant. The equations of motion for such a field are

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right],$$
$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0.$$

Discuss the conditions the field and potential must satisfy at the Planck epoch for 'vacuum domination' and 'slow roll'. Use these conditions to obtain approximate inflationary solutions for $\phi(t)$ and the scale factor, $a(t)$.

Determine the value of $\phi$ when inflation ends, and the overall expansion of the Universe, assuming inflation begins during the Planck era when $\rho \sim M_{Pl}^4$. Calculate the reheat temperature, $T_R$, in terms of $\lambda$ and $g^*$, the effective number of degrees of freedom. Briefly discuss how the horizon problem is solved.

**4** At temperatures above 1MeV the three species of neutrinos and their anti-neutrinos are kept in thermal equilibrium via the interactions $l^- + \nu \leftrightarrow l^- + \nu$, where $l^-$ is a generic lepton. The interaction rate is $\Gamma_\nu \sim G_F^2 T^5$, where $G_F \sim 10^{-5} m_p^{-2}$ and the proton mass is $m_p \sim 1GeV$. At about $T \sim 0.5MeV$ most electrons and positrons annihilate.

Describe in detail why the ratio of the neutrino to photon temperature today is given by

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3} .$$

If one of the neutrino species has a small mass, $m_\nu$, give an estimate of the order of magnitude of this mass by assuming that this massive neutrino is the dominant matter component in a flat FRW universe at the present time. You may take the photon temperature today to be $T_\gamma = 2.73°K = 2 \times 10^{-13} GeV$ and the Hubble time $H_0^{-1} = 10^{10} h^{-1}$ years $= 4.5 \times 10^{41} h^{-1} GeV^{-1}$ with $0.5 < h < 1.0$.

## END OF PAPER