UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 6 June, 2005 9 to 12

PAPER 60

GENERAL RELATIVITY

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

The signature is $(+\,-\,-\,-)$, *and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km}\,.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Information sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write down formulae for the components of the Lie derivative of a scalar field f and a vector field X with respect to a vector field ξ. Show that if there is a metric g_{ab} with a symmetric connection then

$$\mathcal{L}_\xi X = \nabla_\xi X - \nabla_X \xi \,.$$

If X and Y are arbitrary vector fields deduce

$$\left(\mathcal{L}_\xi g\right)_{ab} X^a Y^b = 2\xi_{(a;b)} X^a Y^b.$$

The vector field ξ is said to be *Killing* iff $\mathcal{L}_\xi g_{ab} = 0$. Deduce *Killing's equation*

$$\xi_{(a;b)} = 0 \,.$$

If ξ is Killing, deduce that

$$\xi_{b;ca} = -R_{bca}{}^{d}\xi_d \,.$$

Suppose point P is arbitrary and γ is an arbitrary curve through P with tangent vector V. Setting $L_{ab} = \xi_{a;b}$ deduce that

$$\nabla_V \xi_a = L_{ab} V^b, \qquad \nabla_V L_{ab} = -R_{abc}{}^{d} V^c \xi_d \,.$$

Deduce that if a Killing vector and its first covariant derivative vanish at a point P then they vanish everywhere. In a manifold of dimension n how many linearly independent Killing vectors can there be?

2 Write an essay on the rôle of curvature in general relativity and (stating carefully the assumptions made) the derivation of the Einstein field equations. Your account should pay particular attention to the existence of freely falling frames, the tensorial nature of relative acceleration of neighbouring freely falling particles and the non-vanishing of the Riemann curvature tensor.

3 Consider the linearized Einstein equations for a point mass M at the origin, $\delta T^{ab} = M\delta(\mathbf{x})U^a U^b$ with $U^a = (1, \mathbf{0})$. Obtain the linearized Schwarzschild solution

$$ds^2 = (1 - r_S/r)\, dt^2 - (1 + r_S/r)\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $r_S = 2GM$.

Next consider null geodesics moving, without loss of generality, in the equatorial plane $\theta = \frac{1}{2}\pi$. Setting $u = r_S/r$ show that, within linearized theory,

$$\left(\frac{du}{d\phi}\right)^2 = \frac{r_S^2 E^2}{h^2} - u^2(1-u)\,,$$

where E and h are constants whose physical significance should be described. Investigate whether, for special choices of E/h, circular orbits are possible, and if so discuss their stability.

[*You may use any information from the lecture handout included with this examination paper.*]

4 Let $d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ denote the line element on the unit 2-sphere, and let

$$\widehat{ds}^2 = dt^2 - dr^2 - r^2\, d\Sigma^2$$

denote the Minkowski spacetime line element in spherical polar coordinates. Introduce null coordinates $u = t - r$ and $v = t + r$ and make a conformal transformation to an unphysical spacetime with line element $ds^2 = 4(1+u^2)^{-1}(1+v^2)^{-1}\widehat{ds}^2$. Perform further coordinate changes $p = \tan^{-1} u$, $q = \tan^{-1} v$, followed by $T = q + p$, $R = q - p$ to obtain

$$ds^2 = dT^2 - dR^2 - \sin^2 R\, d\Sigma^2,$$

where the ranges of all of the coordinates t, r, θ, ϕ, u, v, p, q, T and R should be stated explicitly.

Use your results to discuss the asymptotic behaviour of Minkowski spacetime geodesics as seen in the unphysical spacetime. What, if any, horizon structure is there?

LINEARIZED PERTURBATIONS OF MINKOWSKI SPACETIME

In the coordinate chart used here, t is the usual time coordinate of Minkowski spacetime. The spatial coordinates x^α are arbitrary, and the background line element is

$$ds^2 = g_{ik}dx^i dx^k = dt^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta,$$

where $\gamma_{\alpha\beta}$ is a 3-metric of signature $(+++)$, D_α is the metric covariant derivative of $\gamma_{\alpha\beta}$, Δ is the Laplacian and $\Delta_{\alpha\beta} = D_\alpha D_\beta - \frac{1}{3}\gamma_{\alpha\beta}\Delta$.

The full perturbed metric is

$$ds^2 = (1+2A)dt^2 - 2B_\alpha dt dx^\alpha - \big[(1+2C)\gamma_{\alpha\beta} + 2E_{\alpha\beta}\big]dx^\alpha dx^\beta.$$

The left hand side of each equation contain spacetime tensor components and indices are to be raised/lowered using g^{ik}/g_{ik}. When spatial indices occur on the right hand side they should be raised/lowered using $\gamma^{\alpha\beta}/\gamma_{\alpha\beta}$.

Only a minimal set of non-vanishing components is presented. There are other non-vanishing terms which can be generated using the symmetries.

Scalar perturbations in longitudinal gauge

Metric Tensor

$$\delta g_{00} = 2A, \qquad \delta g^{00} = -2A, \qquad \delta g_{\alpha\beta} = -2C\gamma_{\alpha\beta}, \qquad \delta g^{\alpha\beta} = 2C\gamma^{\alpha\beta}.$$

Connection

$$\begin{aligned}
\delta\Gamma^0{}_{00} &= \dot{A}, & \delta\Gamma^0{}_{0\alpha} &= D_\alpha A, & \delta\Gamma^0{}_{\alpha\beta} &= \dot{C}\gamma^{\alpha\beta},\\
\delta\Gamma^\alpha{}_{00} &= D^\alpha A, & \delta\Gamma^\alpha{}_{0\beta} &= \dot{C}\delta^\alpha{}_\beta, & \delta\Gamma^\alpha{}_{\beta\gamma} &= 2\delta^\alpha{}_{(\beta}D_{\gamma)}C - \gamma_{\beta\gamma}D^\alpha C.
\end{aligned}$$

Riemann tensor

$$\begin{aligned}
\delta R^0{}_{\alpha 0\beta} &= [-\ddot{C} + \tfrac{1}{3}\Delta A]\gamma_{\alpha\beta} + \Delta_{\alpha\beta}A,\\
\delta R^o{}_{\alpha\beta\gamma} &= 2\gamma_{\alpha[\beta}D_{\gamma]}\dot{C},\\
\delta R^\alpha{}_{\beta\gamma\delta} &= \tfrac{4}{3}\Delta C\delta^\alpha{}_{[\gamma}\gamma_{\delta]\beta} + 2\delta^\alpha{}_{[\gamma}\Delta_{\delta]\beta} - \gamma_{\beta[\gamma}\Delta_{\delta]}{}^\alpha C.
\end{aligned}$$

Einstein tensor

$$\begin{aligned}
\delta G_{00} &= 2\Delta C,\\
\delta G_{0\alpha} &= 2D_\alpha\dot{C},\\
\delta G_{\alpha\beta} &= \big[2\ddot{C} - \tfrac{2}{3}\Delta(A+C)\big]\gamma_{\alpha\beta} + \Delta_{\alpha\beta}(A+C).
\end{aligned}$$

Vector perturbations in vector gauge

Metric tensor

$$\delta g_{0\alpha} = -B_\alpha, \qquad \delta g^{0\alpha} = -B^\alpha.$$

Connection

$$\delta\Gamma^0{}_{\alpha\beta} = -D_{(\alpha}B_{\beta)}, \qquad \delta\Gamma^\alpha_{00} = \dot{B}^\sigma, \qquad \delta\Gamma^\alpha{}_{0\beta} = \tfrac{1}{2}(D_\beta B^\alpha - D^\alpha B_\beta).$$

Riemann tensor

$$\begin{aligned}\delta R^0{}_{\alpha 0\beta} &= D_{(\alpha}\dot{B}_{\beta)},\\ \delta R^0{}_{\alpha\beta\gamma} &= D_{[\beta}D_{|\alpha|}B_{\gamma]}.\end{aligned}$$

Einstein tensor

$$\begin{aligned}\delta G_{0\alpha} &= \tfrac{1}{2}\Delta B_\alpha,\\ \delta G_{\alpha\beta} &= D_{(\alpha}\dot{B}_{\beta)}.\end{aligned}$$

Tensor perturbations

Metric tensor

$$\delta g_{\alpha\beta} = -2E_{\alpha\beta}, \qquad \delta g^{\alpha\beta} = 2E^{\alpha\beta}.$$

Connection

$$\delta\Gamma^0{}_{\alpha\beta} = \dot{E}_{\alpha\beta}, \qquad \delta\Gamma^\alpha{}_{0\beta} = \dot{E}^\alpha{}_\beta, \qquad \delta\Gamma^a{}_{\beta\gamma} = 2D_{(\beta}E_{\gamma)}{}^\alpha - D^\alpha E_{\beta\gamma}.$$

Riemann tensor

$$\begin{aligned}\delta R^0{}_{\alpha 0\beta} &= -\ddot{E}_{\alpha\beta},\\ \delta R^0{}_{\alpha\beta\gamma} &= -2D_{[\beta}\dot{E}_{\gamma]\alpha},\\ \delta R^\alpha{}_{\beta\gamma\delta} &= -2D_{[\gamma}D_{|\beta|}E_{\delta]}{}^\alpha + 2D_{[\gamma}D^\alpha E_{\delta]\beta}.\end{aligned}$$

Einstein tensor

$$\delta G_{\alpha\beta} = -\ddot{E}_{\alpha\beta} + \Delta E_{\alpha\beta}.$$

END OF PAPER