UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 9 to 12

PAPER 6

INTRODUCTION TO FUNCTIONAL ANALYSIS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) State and prove the Baire category theorem.

(ii) By using the Baire category theorem, or otherwise, show that we can find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} k_j x_j \neq k_{n+1}$$

whenever k_1, k_2, $\ldots$, k_{n+1} are integers, not all of which are zero.

(iii) Show that, given any $K > 0$ we can find a continuous function $f : \mathbb{T} \to \mathbb{R}$ such that $\|f\|_\infty \leqslant 1$ but the N-th partial Fourier sum $S_N(f, 0)$ satisfies

$$|S_N(f, 0)| > K$$

for some N.

(iv) Show that there exists a continuous function whose Fourier series diverges at 0.

2 Let X be a real vector space and $p,\ q : X \to \mathbb{R}$ be functions such that $p(\lambda x) = \lambda p(x)$, $q(\lambda x) = \lambda q(x)$ for all $\lambda \in \mathbb{R}$ with $\lambda \geqslant 0$ and all $x \in X$, whilst

$$p(x+y) \leqslant p(x) + p(y),\ \ q(x) + q(y) \leqslant q(x+y)$$

for all $x,\ y \in X$.

(i) Suppose that Y is a subspace of X and $S : Y \to \mathbb{R}$ a linear function such that

$$S(y) \leqslant p(x+y) - q(x)$$

for all $x \in X,\ y \in Y$. Show that

$$S(y') - p(x' + y' - z) + q(x') \leqslant -S(y) + p(x+y+z) - q(x)$$

for all $x,\ x',\ z \in X$ and $y,\ y' \in Y$.

(ii) Suppose that Y_0 is a subspace of X and $T_0 : Y_0 \to \mathbb{R}$ a linear function such that

$$T_0(y) \leqslant p(x+y) - q(x)$$

for all $x \in X,\ y \in Y_0$. Show that there exists a linear function $T : X \to \mathbb{R}$ such that

$$T(y) \leqslant p(x+y) - q(x)$$

for all $x,\ y \in X$ and $Tu = T_0 u$ for all $u \in Y_0$. Show that

$$q(x) \leqslant T(x) \leqslant p(x)$$

for all $x \in X$.

(iii) Suppose that $p(x) \geqslant q(x)$ for all $x \in X$. Show that there exists a linear function (possibly the zero function) $U : X \to \mathbb{R}$ such that

$$q(x) \leqslant U(x) \leqslant p(x)$$

for all $x \in X$.

3 Let B be a commutative Banach algebra with a unit. Develop the theory of the resolvent of an element $x \in B$ up to and including the formula

$$\rho(x) = \sup\{|\lambda| : \lambda e - x \text{ is not invertible}\}$$

for the spectral radius.

Give an example of a B and an $x \in B$ for which $\rho(x) = 0$ although $x \neq 0$. Give an example of a B and an $x \in B$ for which $\rho(x) = \|x\|_B = 1$.

[You may assume results from the theory of vector valued integration but not from the theory of Banach algebra valued analytic functions.]

4 Let $C([-1,1])$ be the space of real valued continuous functions on $[-1,1]$ under the uniform norm. Consider the subspace $\mathcal{P}_n$ of real polynomials of degree at most n. You may assume that, if T is a linear map $\mathcal{P}_n \to \mathbb{R}$, with $\|T\| = 1$, then, given $\epsilon > 0$, we can find an $N \geqslant 1$ and $\lambda_1, \lambda_2, \ldots \lambda_N \in \mathbb{R}$ with $\sum_{j=1}^N |\lambda_j| = 1$ and $x_1, x_2, \ldots x_N \in [-1,1]$ such that

$$\left| TP - \sum_{j=1}^N \lambda_j P(x_j) \right| < \epsilon.$$

Show, proving the results (such as Caratheory's theorem) that you need, that, if P is a real polynomial of degree at most n and $u \notin [-1,1]$, then

$$|P(u)| \leqslant \sup_{x \in [-1,1]} |P(x)||T_n(u)|$$

where T_n is the Tchebychev polynomial of degree n.

END OF PAPER