UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 1.30 to 3.30

PAPER 58

INTRODUCTION TO QUANTUM COMPUTATION

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

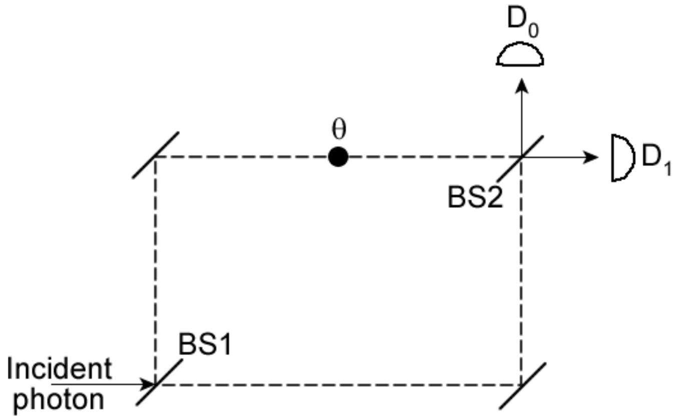
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the following experiment, where BS1 and BS2 are 50:50 beam splitters and D_0 and D_1 are photon detectors:



When passing through the upper arm of the interferometer, a photon gains a (tunable) phase of θ relative to its passage through the lower arm.

(i) What is the probability, as a function of θ, that the incident photon is detected at D_1?

(ii) If we replaced BS1 with a polarising beam splitter (so horizontal polarisations are transmitted and vertical polarisations are reflected), and then added an element to the top arm of the interferometer that rotates the vertical component to horizontal, how can we determine the polarisation state of a source of photons (all with the same polarisation)?

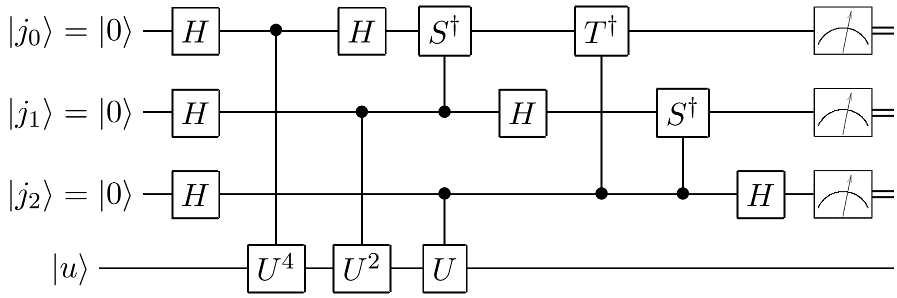
(iii) Using the apparatus already described, is there an easier way that we could have done this?

(iv) What would have happened in part (ii) if we had not inserted the element that rotated the vertical polarisation to horizontal? Justify your answer with a short calculation.

2 (i) Show that the phase estimation procedure acting on t qubits (shown in the diagram below for $t = 3$) can find the phase of the eigenvalue of an eigenvector, $|u\rangle$, of the unitary operator U, perfectly if the phase is of the form

$$\phi = \frac{2\pi n}{2^t}$$

where n is a positive integer.

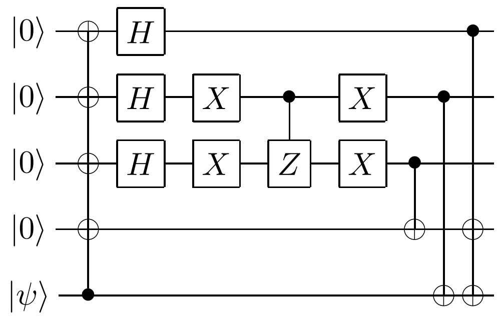


$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

(ii) If the phase is within δ of one of these perfect solutions ($\delta \ll \frac{2\pi}{2^t}$), show what result phase estimation gives.

(iii) Describe a potential application for phase estimation and why it is of interest.

3 A qubit is transmitted through a channel and experiences an error with probability p. In an attempt to reduce the error, we choose to encode the state across 5 qubits using the encoding circuit shown below. This error correcting code can correct for an error on, at most, one qubit. If there is more than one error, this propagates to all the encoded qubits.



(i) How is the input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ encoded?

(ii) Given that all 5 qubits are now transmitted through the error-prone channel, what is the probability of correct recovery of the state? Give an expression for the threshold below which it makes more sense to encode the state in this form.

(iii) Consider encoding the code, so that each of the 5 qubits is now encoded across 5. After transmission of all 25 qubits through the channel, we error correct each block of 5 qubits, and the output acts like a single input to a second, higher level of error correction. What advantages might this have?

4 Two common distance measures between the quantum states (density matrices) ρ_1 and ρ_2 are the trace distance,

$$D(\rho_1, \rho_2) = \tfrac{1}{2}\mathrm{Tr}\ \sqrt{(\rho_1 - \rho_2)^\dagger(\rho_1 - \rho_2)}$$

and the fidelity,

$$F(\rho_1, \rho_2) = \mathrm{Tr}\ \sqrt{\rho_1^{1/2}\rho_2\,\rho_1^{1/2}}\ .$$

(i) Using the fact that for any positive operator, A,

$$\sqrt{UAU^\dagger} = U\sqrt{A}U^\dagger,$$

show that the two distance measures are invariant under unitary transformations, U, of the quantum states.

(ii) Use this result, in the case that ρ_1 and ρ_2 are both pure states, to give the trace distance in terms of the fidelity.

(iii) If a single qubit, initially in a pure state $\rho_1 = |\psi\rangle\langle\psi|$, experiences some decoherence which results in a bit flip with probability p, what is its final density matrix, ρ_2? What is the average of the square of the fidelity, over all possible states, of the decohered state (i.e. how well has it survived)? You may wish to use the parameterisation $|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$.

END OF PAPER