UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 9 to 12

PAPER 52

ADVANCED QUANTUM FIELD THEORY

Attempt **THREE** *questions.*

There are **FOUR** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The asymptotic condition for an interacting hermitian quantum field $\phi(x)$ is

$$\phi(x) \to \sqrt{Z}\phi_{\text{in}}(x) \qquad \text{as} \qquad x^0 \to -\infty \ ,$$

where $\phi_{\text{in}}(x)$ is a free field that obeys $(\partial^2 + m^2)\phi_{\text{in}}(x) = 0$. State the corresponding condition for the limit $x^0 \to +\infty$. Given an appropriate choice of phases for states show that

$$\sqrt{Z} = \langle p|\phi(0)|0\rangle \ ,$$

where $|0\rangle$ is the vacuum state and $|p\rangle$ is the single particle state with 4-momentum p .

Let $\{|\alpha\rangle\}$ be a complete set of eigenstates of the 4-momentum operator P_μ, that is

$$P_\mu|\alpha\rangle = p_\mu^{(\alpha)}|\alpha\rangle \ ,$$

and let the propagator for the field $\phi(x)$ be $\Delta(x)$ where

$$i\Delta(x) = \langle 0|T\phi(x)\phi(0)|0\rangle \ .$$

Show that

$$i\Delta(x) = \sum_\alpha |\langle\alpha|\phi(0)|0\rangle|^2 \left[\theta(x^0)e^{-ip^{(\alpha)}.x} + \theta(-x^0)e^{ip^{(\alpha)}.x}\right] \ ,$$

and hence that

$$\Delta(x) = \int_0^\infty d\sigma \rho(\sigma)\Delta_F(x,\sigma) \ ,$$

where $\Delta_F(x,\sigma)$ is the Feynman propagator for a free scalar field of mass $\sqrt{\sigma}$ and $\rho(\sigma)$ is a quantity you should define. Show that the single particle contribution to $\rho(\sigma)$ is

$$\rho(\sigma) = Z\delta(\sigma - m^2) \ ,$$

where m is the mass of the single particle state. Show that

$$\int_0^\infty d\sigma \rho(\sigma) = 1 \ ,$$

and obtain a restriction on the value of Z .

2 The dynamics of a hermitian quantum scalar field $\phi(x)$ is determined by the Lagrangian $\mathcal{L}(x)$ where

$$\mathcal{L}(x) = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m_0^2\left(\phi\right)^2 - \frac{\lambda_0}{4!}\left(\phi\right)^4 \quad .$$

Write down the Feynman rules for the amplitude associated with a perturbation theory diagram.

Let the diagram have N_v vertices, N_l internal lines and N_e external lines. Define the superficial degree of divergence, D, of the diagram and use the graph theory result

$$4N_v - N_e = 2N_l$$

to show that in four dimensions

$$D = 4 - N_e \quad .$$

What is the significance of this result for the renormalisability of ϕ^4-theory in four dimensions? Draw Feynman diagrams for the lowest order contributions to the two point and four point functions. Check your general result for D in these cases. Draw a one loop graph that contributes to the six point function and demonstrate that it introduces no new divergences into the theory.

Explain why the mass of the particle in the theory differs from the bare mass m_0 and show that to $O(\lambda_0)$ the physical mass becomes m where

$$m^2 = m_0^2 + \frac{\lambda_0}{2(4\pi)^2}\left[\Lambda^2 - m_0^2 \log\left(\frac{\Lambda^2 + m_0^2}{m_0^2}\right)\right]$$

where Λ is an ultraviolet cutoff introduced after a Wick contour rotation.

3 The method of dimensional regularisation requires a quantum field theory to be formulated in n dimensions of space-time. By exhibiting the relevant Feynman diagrams, but without performing a detailed calculation, explain why in ϕ^4-theory the bare coupling λ_0 is related to a renormalised coupling λ_R by the formula

$$\lambda_0 = (\mu^2)^{2-\frac{n}{2}} \left(\lambda_R + \lambda_R^2 \frac{C}{(n-4)} + \dots \right) ,$$

where μ is a mass scale parameter. You may assume that

$$C = -\frac{3}{16\pi^2} .$$

Show that if μ is varied but the theory remains unchanged then the renormalised coupling must change in such a way that

$$\mu \frac{d\lambda_R}{d\mu} = \beta(\lambda_R) ,$$

where, to $O(\lambda_R^2)$,

$$\beta(\lambda_R) = -(4-n)\lambda_R + \frac{3}{16\pi^2}\lambda_R^2 .$$

Using this, show that in four dimensions

$$\lambda_R(\mu) = \lambda_s \left(1 - \frac{3\lambda_s}{16\pi^2} \log \frac{\mu}{\mu_s} \right)^{-1} ,$$

where λ_s is the value of the renormalised coupling when $\mu = \mu_s$.

Consider the situation where a more detailed knowledge of the function $\beta(\lambda_R)$ reveals that it has a zero at $\lambda_R = \lambda^*$. Explain the ideas of ultraviolet and infrared stability in relation to the fixed point λ^* . Show that when $n < 4$ the theory has an infrared stable fixed point at

$$\lambda_R = (4-n)\frac{16\pi^2}{3} .$$

What does this imply about the interactions of the particles of the theory when $n = 4$?

4 Let G be a simple Lie group of $N \times N$ unitary matrices $\{h_{ab}\}$ with hermitian generators $\{T_\alpha\}$ $\alpha = 1, 2, \ldots, M$. You may assume that for h an element of G

$$h = \exp\{-i\theta_\alpha T_\alpha\} ,$$

for parameter values $\{\theta_\alpha\}$ and that the generators satisfy the Lie algebra

$$[T_\alpha, T_\beta] = i f_{\alpha\beta\gamma} T_\gamma .$$

A scalar field multiplet $\phi = \{\phi_a\}$ $a = 1, 2, \ldots, N$ transforms under G . Verify that the Lagrangian for the theory

$$\mathcal{L}_\phi = \partial^\mu \phi^\dagger \partial_\mu \phi - m_0^2 \phi^\dagger \phi - \lambda_0 \left(\phi^\dagger \phi\right)^2$$

is invariant under the action of the group G .

Show how, by introducing a gauge field $A_\mu(x) = A_{\alpha\mu}(x) T_\alpha$ whose transformation properties you should clearly specify, the theory can be modified so that it is *gauge invariant*, that is invariant under *local* group transformations $\phi(x) \to h(x)\phi(x)$.

Use the functional integral formalism to show how to calculate the Green's functions for gauge fields and explain the need for gauge fixing. Show how the introduction of a gauge fixing term into the Lagrangian requires also the introduction of anti-commuting ghost fields $c_\alpha(x)$ and $\bar{c}_\alpha(x)$.