UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 14 June, 2005 9 to 11

PAPER 51

STATISTICAL FIELD THEORY

Attempt ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain briefly what is meant by a phase diagram. Give an example of a three-dimensional phase diagram which contains a tricritical point and describe the nature of the different phase transitions which can occur.

Give an account of the Landau-Ginsberg (LG) theory of phase transitions illustrated using scalar field theory including a discussion of the following topics:

(a) The idea of an order parameter;

(b) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(c) The idea of universality and which properties are, and are not, universal at a critical point;

(d) The reason why the ferromagnet and H_2O show critical behaviours that belong to the same universality class;

(e) The idea of *critical exponents* and how they may be derived;

(f) The features of a tricritical point, how it occurs in LG theory and a brief description of a phase diagram containing a tricritical point;

(g) The notion of critical dimension D_c and why predictions for critical exponents in LG theory fail for $D \leq D_c$.

You should clarify your account with diagrams which should be appropriately labelled.

For a system described by a single scalar field the critical indices α and δ are defined by

$$C \;=\; T\frac{\partial^2 A}{\partial T^2} \;\sim\; |t|^{-\alpha}\;, \quad (h=0) \qquad \text{and} \qquad M \;\sim\; h^{1/\delta}\;, \quad (t=0)\;,$$

where $t = (T - T_c)/T_c$, C is the specific heat and A is the appropriate free energy. Using LG theory calculate α and δ both for an ordinary critical point and a tricritical point.

2 A D-dimensional cubical lattice Λ can be described as a checker-board of black (B) and white (W) sites such that all nearest neighbours of black sites are white, and vice-versa. The lattices of black and white sites are denoted by Λ_B and Λ_W, respectively, so that $\Lambda = \Lambda_B + \Lambda_W$.

A model defined on Λ has Hamiltonian

$$\mathcal{H} \;=\; -h \sum_{\mathbf{n}\in\Lambda} \sigma_{\mathbf{n}} - g \sum_{\mathbf{b}\in\Lambda_B} \sigma_{\mathbf{b}} + g \sum_{\mathbf{w}\in\Lambda_W} \sigma_{\mathbf{w}} \;,$$

where $\sigma_{\mathbf{n}} \in \{1, -1\}$, h is the value of the magnetic field and g is the value of the staggered magnetic field. Show that the magnetization per site, M_B and M_W, for black and white sites respectively, are given by

$$\begin{aligned} M_B = \langle \sigma_{\mathbf{b}} \rangle \;&=\; \tanh(\beta(h+g)) \;\; \mathbf{b} \in \Lambda_B \;, \\ M_W = \langle \sigma_{\mathbf{w}} \rangle \;&=\; \tanh(\beta(h-g)) \;\; \mathbf{w} \in \Lambda_W \;. \end{aligned}$$

Explain why, when $h = 0$, you expect the result $M_W = -M_B$ given by these equations.

The Hamiltonian for the antiferromagnetic Ising model in D-dimensions is given by

$$\mathcal{H} \;=\; J \sum_{\mathbf{n}\in\Lambda,\mu} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\mu} - h \sum_{\mathbf{n}\in\Lambda} \sigma_{\mathbf{n}} - g \sum_{\mathbf{b}\in\Lambda_B} \sigma_{\mathbf{b}} + g \sum_{\mathbf{w}\in\Lambda_W} \sigma_{\mathbf{w}} \;.$$

where μ is summed over the basis vectors of the unit cell of Λ. In the Mean Field Approximation we employ the following identity for the product of given nearest neighbour spins, $\sigma_{\mathbf{b}}$ and $\sigma_{\mathbf{w}}$:

$$\sigma_{\mathbf{b}} \sigma_{\mathbf{w}} \;=\; (M_B + (\sigma_{\mathbf{b}} - M_B))(M_W + (\sigma_{\mathbf{w}} - M_W)) \;.$$

Briefly explain the meaning of this relation in the context of the Mean Field Approximation, including why you expect it to be useful only when D is sufficiently large.

Show that the mean field Hamiltonian is

$$\mathcal{H}_{MF} \;=\; -\tfrac{1}{2} q J M_B M_W - f_B \sum_{\mathbf{b}\in\Lambda_B} \sigma_{\mathbf{b}} - f_W \sum_{\mathbf{w}\in\Lambda_W} \sigma_{\mathbf{w}} \;,$$

where $q = 2D$, and where f_B and f_W are to be determined. Hence, show that

$$M_B \;=\; \tanh(\beta f_B) \;, \qquad M_W \;=\; \tanh(\beta f_W) \;. \qquad (*)$$

When $h = 0$ show that there are solutions to these equations of state with $M_B = -M_W \equiv M$, and show a graphical representation of these solutions. Hence, show that there is a second-order phase transition with order parameter $M_{-} = \frac{1}{2}(M_B - M_W)$ at $T = T_c = qJ/k$, where k is Boltzmann's constant.

Question 2 continues on the next page

Show that when $g = 0$ there is just one solution for $M_B = M_W \equiv M \; \forall T$, and hence that there is no phase transition in the order parameter $M_+ = 1/2(M_B + M_W)$. Explain why this should be the case.

Derive an expression for the free energy per site, $A(M_B, M_W)$, and verify the equations of state $(*)$ above. Expand $A(M_B, M_W)$ to second order in M_+, M_- and to first order in h and g. Confirm that there is a second-order phase transition in M_- and verify the value for T_c previously obtained.

Some relevant susceptibilities are defined by

$$\chi_B \; = \; \frac{\partial M_B}{\partial h} \, , \quad \chi_W \; = \; \frac{\partial M_W}{\partial h} \, , \quad \chi_\pm \; = \; 1/2(\chi_B \pm \chi_W) \, .$$

Show that for $h = g = 0$ and $T < T_c$

$$\chi_+ \; = \; \frac{\beta(1 + M^2)}{1 + \beta qJ + \beta qJM^2} \, , \quad \chi_- \; = \; 0 \, .$$

Why is the result for χ_- to be expected?

What are the results for $\chi_\pm$ when $T > T_c$?

3 The Ising model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$ and $\sigma_r \;\in\; \{1,-1\}$. In particular, $\mathcal{H}$ contains the term $-h\sum_r \sigma_r$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u},C,N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u},\sigma) - \beta NC) \;.$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour for $r \gg \xi$.

State how the magnetization M and the magnetic susceptibility χ can be determined from the free energy F, and derive the relation which expresses χ in terms of $G(\mathbf{r})$.

Give a short account of how the idea of the Real Space Renormalization Group can be applied to this model including a discussion of the following topics:

(a) The idea of a blocking kernel;

(b) The rôle of fixed points and the scaling hypothesis;

(c) The idea of *relevant* and *irrelevant* operators;

(d) The separation of the expression for the free energy into a singular part $f(\mathbf{u})$ and an inhomogeneous part depending on a function $g(\mathbf{u})$ whose rôle should be defined;

(e) The reason why the inhomogeneous contribution to the free energy may generally be ignored when computing critical exponents.

(f) The derivation of critical exponents in terms of the relevant scaling exponents.

The Gaussian model in D dimensions ($D \le 4$) for a real scalar field is defined by the Hamiltonian

$$\mathcal{H} \;=\; \tfrac{1}{2}\left(\kappa^{-1}\left(\nabla\phi(\mathbf{x})\right)^2 + m^2\phi^2(\mathbf{x})\right) + h\phi(\mathbf{x}) \;,$$

where κ, m and h are coupling constants.

Derive an expression for the correlation length ξ in terms of the coupling constants.

By defining a suitable thinning transformation show that the critical exponents α and β are given by

$$\alpha \;=\; (4-D)/2 \;, \qquad \beta \;=\; (D-2)/4 \;.$$

END OF PAPER