UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 12

PAPER 5

SOME INEQUALITIES

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State Hölder's inequality.

Suppose that $1/p_1 + \cdots + 1/p_n = 1/r \leqslant 1$ and that $f_i \in L_{p_i}$ for $1 \leqslant i \leqslant n$. Show that $f_1 \ldots f_n \in L_r$ and

$$||f_1 \ldots f_n||_r \leqslant ||f_1||_{p_1} \ldots ||f_n||_{p_n} .$$

Suppose that K is a compact subset of $\mathbf{R}^4$. Let K_j be the image of K under the orthogonal projection onto the subspace orthogonal to the j-th axis. Show that

$$\lambda_4(K) \leqslant (\prod_{j=1}^{4} \lambda_3(K_j))^{1/3}.$$

[Here λ_d denotes d-dimensional Lebesgue measure.]

[You may assume the truth of the corresponding result, and related results, in dimension 3.]

2 Suppose that $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ are decreasing sequences of positive numbers, that $\sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} y_i$ for each n and that $\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} y_i$. Show that there exists a doubly stochastic matrix $P = (p_{ij})$ such that $x_i = \sum_{j=1}^{\infty} p_{ij} y_j$, for each i.

3 (i) What does it mean to say that a sublinear mapping S of $L^1(\mathbf{R})$ into $M(\mathbf{R})$ (the measurable functions on $\mathbf{R}$) is of weak type $(1,1)$?

(ii) Suppose that $(T_r)_{r \geqslant 0}$ is a family of linear mappings from $L^1(\mathbf{R})$ into $M(\mathbf{R})$ and that S is a sublinear mapping of $L^1(\mathbf{R})$ into $M(\mathbf{R})$ which is of weak type $(1,1)$, such that

(a) $|T_r(g)| \leqslant S(g)$ for all $g \in L^1(\mathbf{R})$, $r \geqslant 0$, and

(b) there is a dense subspace F of E such that $T_r(f) \to T_0(f)$ almost everywhere, for $f \in F$, as $r \to 0$.

Show that if $g \in E$ then $T_r(g) \to T_0(g)$ almost everywhere, as $r \to 0$.

(iii) If $f \in L^1(\mathbf{R})$, let

$$m_u(f)(x) = \sup\{\frac{1}{r}\int_y^{y+r} |f(t)|\, dt : r > 0,\ y < x < y + r\}.$$

Show that m_u is of weak type $(1,1)$. [You may quote any covering lemma that you need.]

(iv) Suppose that $f \in L^1(\mathbf{R})$. Let $F(x) = \int_0^x f(t)\, dt$. Show that F is differentiable almost everywhere.

4 (i) Suppose that $1 \leqslant p_0 < p_1 < \infty$, that $0 < \theta < 1$ and that $1/p = (1-\theta)/p_0 + \theta/p_1$. Show that

$$L^{p_0} \cap L^{p_1} \subseteq L^p \subseteq L^{p_0} + L^{p_1}$$

and that if $f \in L^p$ then we can write $f = g + h$ with $\|g\|_{p_0}^{1-\theta} \|h\|_{p_1}^{\theta} \leqslant \|f\|_p$.

(ii) Suppose further that $1 \leqslant q_0, q_1 \leqslant \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. In the proof of the Riesz-Thorin Theorem it is shown that if T is a linear mapping from $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$ with $||T : L^{p_i} \to L^{q_i}|| = M_i$, for $i = 0, 1$, then if f is a simple function, $||T(f)||_q \leqslant M_0^{1-\theta} M_1^{\theta} ||f||_p$. Show how this result extends to any $f \in L^p$.

(iii) Suppose that $2 < p < \infty$ and that $1/p + 1/p' = 1$. Show that if x and y are complex numbers then

$$\left(\tfrac{1}{2}(|x+y|^p + |x-y|^p)\right)^{1/p} \leqslant \left(|x|^{p'} + |y|^{p'}\right)^{1/p'}.$$

Show further that if f and g are in L^p then

$$\left(\tfrac{1}{2}(\|f+g\|_p^p + \|f-g\|_p^p)\right)^{1/p} \leqslant \left(\|f\|_p^{p'} + \|g\|_p^{p'}\right)^{1/p'}.$$

[Hint: Use Minkowski's inequality in $L^{p/p'}$.]

5 Suppose that $a_1, \ldots, a_d$ are vectors in a normed space E and that $\epsilon_1, \ldots, \epsilon_d$ are independent Bernoulli random variables. Show that

$$\left\| \sum_{i=1}^{d} \epsilon_i a_i \right\|_{L^2(E)} \leqslant \sqrt{2} \left\| \sum_{i=1}^{d} \epsilon_i a_i \right\|_{L^1(E)}.$$

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