UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 9 to 12

# PAPER 49

# QUANTUM FIELD THEORY

*Attempt* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** The free Klein-Gordon field $\phi(\mathbf{x}, t)$ obeys the equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \, .$$

Using Noether's theorem, find the expression for the conserved 3-momentum $\mathbf{P}$.

In the quantized Klein-Gordon theory, the field $\phi(\mathbf{x})$ and the conjugate field $\pi(\mathbf{x})$ (in the Schrödinger representation) have the coupled expansions

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger \, e^{-i\mathbf{p}\cdot\mathbf{x}})$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger \, e^{-i\mathbf{p}\cdot\mathbf{x}})$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}\cdot\mathbf{p} + m^2}$ and $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ satisfy

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a_{\mathbf{p}}^\dagger \, , \; a_{\mathbf{p}'}^\dagger] = 0$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \, .$$

Show that the 3-momentum operator $\mathbf{P}$ in the quantized theory can be expressed as

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \, \mathbf{p} \, a_{\mathbf{p}}^\dagger \, a_{\mathbf{p}} \, .$$

Calculate $[\mathbf{P}, a_{\mathbf{q}}^\dagger]$ and hence determine

$$e^{-i\mathbf{P}\cdot\mathbf{y}} a_{\mathbf{q}}^\dagger \, e^{i\mathbf{P}\cdot\mathbf{y}}$$

where $\mathbf{y}$ is a constant vector. What can you deduce about $e^{-i\mathbf{P}\cdot\mathbf{y}}|\mathbf{q}\rangle$, where $|\mathbf{q}\rangle$ denotes the one-particle state of 3-momentum $\mathbf{q}$?

Find $e^{-i\mathbf{P}\cdot\mathbf{y}}\phi(\mathbf{x})e^{i\mathbf{P}\cdot\mathbf{y}}$, and interpret your result.

**2** In the chiral representation, the Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The matrix $\gamma^5$ is defined by

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 .$$

Calculate $\gamma^5$ and show that it anticommutes with $\gamma^0$ and $\gamma^i$.

Consider the massless Dirac equation

$$i\gamma^\mu\partial_\mu\psi = 0$$

for a left-handed spinor field, satisfying $\gamma^5\psi = -\psi$. A plane wave solution is of the form

$$\psi(x) = \lambda(p)e^{-ip\cdot x} .$$

What condition does the 4-momentum $p^\mu$ have to satisfy for such a solution to exist? Find and solve the equation for $\lambda(p)$, assuming the condition is satisfied.

Find the effect on this solution of a spatial rotation around the axis parallel to the 3-momentum $\mathbf{p}$. What can you deduce about the spin states of particles in the quantized theory of a left-handed Dirac field?

**3** The interaction picture field $\phi(x)$ of a quantized scalar field theory satisfies the relation

$$\mathrm{T}\phi(x)\phi(y) = \ :\phi(x)\phi(y): \ + \ D_F(x-y)\,.$$

Explain the meaning of the various expressions occurring here, and establish this relation.

Explain in outline how you would derive the Feynman rules for correlation functions

$$\langle 0|\mathrm{T}(\,\phi(x_1)\phi(x_2)\ldots\phi(x_n)S\,)|0\rangle\,,$$

where $S$ is the $S$-matrix, in the theory whose Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{\mu}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4\,.$$

For what range of parameter values do you expect this theory to have a stable vacuum?

$$\left[\ \phi(x)\ \ \textit{has the expansion}\ \ \phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x})\,.\ \right]$$

**4** Write notes on:

(a) Gauge invariance of a classical electromagnetic field coupled to matter fields.

(b) The effect of position-independent gauge transformations, $\psi(x) \to e^{ie\chi}\psi(x)$, on the Feynman rules for QED scattering amplitudes, and the relationship with charge conservation.

(c) Gauge invariance and the photon propagator.

(d) The Landau gauge $\partial_\mu A^\mu = 0$, and the photon propagator in Landau gauge

$$-\frac{i}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\,.$$

## END OF PAPER