UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 12

PAPER 40

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Use the dual simplex algorithm to solve the problem:

$$\text{minimize} \quad 2x_1 + 15x_2 + 18x_3$$

subject to

$$\begin{array}{rrrl} -x_1 + & 2x_2 - & 6x_3 & \leqslant -10 \\ & x_2 + & 2x_3 & \leqslant 6 \\ 2x_1 & + & 10x_3 & \leqslant 19 \\ -x_1 + & x_2 & & \leqslant -2 \end{array}$$

$$x_1, x_2, x_3 \geqslant 0\,.$$

Now use Gomory's cutting plane method to solve the problem when x_1, x_2, x_3 must be integers.

2 Let FP denote the feasibility problem: 'Is the set $P = \{x : Ax \geqslant b,\, x \in \mathbb{R}^n\}$ nonempty?' Here A is a $m \times n$ matrix, where $m \geqslant n$, and the components of A and b are integers with absolute values no more than U. How many bits are needed to state an instance of FP?

Show that if P is nonempty then there exists $x \in P$ such that each component of x can be written as the quotient of two integers, each of which is in absolute value no more than $(nU)^n$.

Deduce that FP is in complexity class $\mathcal{NP}$.

Assuming that the ellipsoid algorithm can solve FP in polynomial time, prove that there exists a polynomial-time algorithm for the problem: minimize $c^\top x$, $Ax \geqslant b$.

3 State and prove Nash's theorem concerning the existence of an equilibrium in a two-person non-zero-sum game. You may assume the Brouwer Fixed Point Theorem.

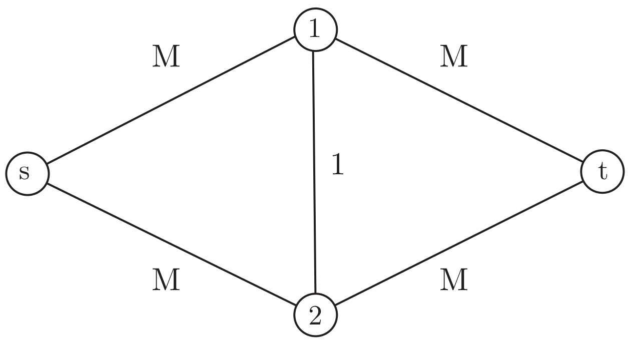
4 State a formula for the Shapley values of a coalitional game. What axioms do they satisfy?

Show that if each player receives a payoff equal to his Shapley value then it is true to say: 'The payoff I lose if you leave the game is equal to the payoff you lose if I leave the game.'

Suppose agent i knows about a set of books B_i. If a set of agents S pool what they know then their payoff is the number of books about which they collectively know, i.e., $\left|\bigcup_{i \in S} B_i\right|$. Show that the game is superadditive and the core is nonempty only if the sets $B_1, \ldots, B_n$ are disjoint.

Show that agent i has Shapley value $x_i = \sum_{b \in B_i} \left|\{k : b \in B_k\}\right|^{-1}$.

5 Consider the undirected graph below, with integer-valued capacities marked on its edges. It is desired to find the maximum flow between s and t. Show that, depending on choices made, the Ford-Fulkerson algorithm might take as few as 2 or as many as $M + 1$ steps to terminate.



Suppose that in a certain undirected graph G with integer-valued edge capacities (c_{ij}) the maximum possible flow between nodes s and t is f^*. Let (x_{ij}) be a feasible flow that sends flow of f from s to t, where x_{ij} is the flow from i to j along edge $\{i, j\}$. Let the 'residual graph' be obtained by supposing edges are directed and the capacities are set to $c'_{ij} = c_{ij} - x_{ij} + x_{ji}$. By using the fact that the minimum cut equals maximum flow show that the maximal flow possible between s and t in the residual graph is $f' = f^* - f$.

Let m be the number of edges of G. Let U be the set of nodes in the residual graph that can be connected to s by a path of capacity of at least f'/m. Show that $t \in U$.

A modified Ford-Fulkerson algorithm adds the rule that whenever flow might be increased on more than one path from s to t then we choose a path on which the greatest increase can be made. Show that after k steps of this algorithm the maximal flow possible from s to t in the residual graph is no more than $(1 - 1/m)^k f^*$.

Hence prove that this algorithm terminates in $O\big(m \log(f^*)\big)$ steps.

6 In a certain a sealed-bid auction bidders compete for a single item. The winner is the highest bidder and he pays the second highest bid. Show that it is a Nash equilibrium for each bidder to bid his valuation.

Explain what is meant by an auction with symmetric independent private values (SIPV).

Suppose a SIPV auction has n bidders whose valuations are uniformly distributed on $[0, 1]$. Show that if the winner must pay his own bid then it is not a Nash equilibrium for bidders to bid their valuations, but that it is a Nash equilibrium for them to bid $(n-1)/n$ times their valuations.

END OF PAPER