UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 1.30 to 4.30

PAPER 4

GROUPS OF LIE TYPE

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let F_{q^2} be a finite field with q^2 elements and $^-$ be the non-trivial automorphism of F_{q^2}, which fixes its subfield F_q pointwise. Let $V = F_{q^2}^n$, $n \geq 2$, and let $B = (\ ,\ )$ be a non-degenerate unitary form on V. Show that there exists an orthonormal basis of V with respect to B. Define the unitary group $U_n(q^2)$.

(ii) Show that V has a basis

$$e_1, \ldots, e_m, f_1, \ldots, f_m, \text{ if } n = 2m,$$

$$e_1, \ldots, e_m, f_1, \ldots, f_m, d \text{ if } n = 2m+1,$$

such that $(e_i, e_j) = (f_i, f_j) = (e_i, d) = (f_i, d) = 0$ for all i, j; $(e_i, f_j) = 0$ for $i \neq j$, $(d, d) = 1$, and $(e_i, f_i) = 1$ for all i. (Hint: Let $v_1, \ldots, v_n$ be an orthonormal basis and let ζ be a root of the polynomial $x^2 - x - 1$. Consider $e_1 = v_1 + \zeta v_2$, $f_1 = \zeta v_1 + v_2$.)

2 (i) Let Φ be a system of roots. For $r, s \in \Phi$, let θ_{rs} be the angle between r and s. Show that θ_{rs} is one of 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$, π. For each value provide an example of a root system, which contains a pair of roots with the given angle.

(ii) Assume that r and s are not orthogonal. What can you say about their relative lengths?

(iii) Show that any root system of rank 2 is equivalent to one of the following systems: $A_1 + A_1$, A_2, B_2, G_2.

(iv) Explain what is meant by the Dynkin diagram of a root system. Write briefly on the classification of indecomposable root systems. Include the corresponding Dynkin diagrams in your answer.

3 Let K be a field.

(i) Prove that $SL_2(K)$ is generated by matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},$$

where t, s run through K.

(ii) Prove that the Chevalley group $A_1(K)$ is isomorphic to $PSL_2(K)$. Give a detailed proof for $K = \mathbf{C}$ and outline briefly the argument for a general field.

4 Let $\mathcal{L}$ be a simple complex Lie algebra.

(i) For each element x of $\mathcal{L}$ define a map $\operatorname{ad} x : \mathcal{L} \to \mathcal{L}$ by $(\operatorname{ad} x)(y) = [x, y]$. Show that $\operatorname{ad} x$ is a derivation of $\mathcal{L}$.

(ii) Assume that $\operatorname{ad} x$ is nilpotent. Give the definition of the map $\exp(\operatorname{ad} x)$ and show that $\exp(\operatorname{ad} x)$ is an automorphism of $\mathcal{L}$.

(iii) Assume that $\operatorname{ad} x$ is nilpotent. Let θ be an automorphism of $\mathcal{L}$. Show that

$$\theta \exp(\operatorname{ad} x)\theta^{-1} = \exp(\operatorname{ad} \theta x).$$

(iv) Show that $\operatorname{ad} x.\operatorname{ad} y - \operatorname{ad} y.\operatorname{ad} x = \operatorname{ad}\,[x, y]$.

(v) Let $\{h_s, e_r : s \in \Pi, r \in \Phi\}$ be a Chevalley basis of $\mathcal{L}$. For a complex number t, let $x_r(t) = \exp(t \operatorname{ad} e_r)$. Let r and s be two linearly independent roots such that $r + s$ is not a root. Show that

$$x_s(u)^{-1}x_r(t)^{-1}x_s(u)x_r(t) = 1.$$

(vi) Assume that all roots in Φ are of the same length. Let r and s be two roots such that $r + s$ is also a root. Define $N_{r,s}$ by $[e_r, e_s] = N_{r,s}e_{r+s}$. Prove the following special case of the Chevalley commutator formula:

$$x_s(u)^{-1}x_r(t)^{-1}x_s(u)x_r(t) = x_{r+s}(-N_{r,s}tu).$$

5 (i) Let A be an $n \times n$ matrix over $\mathbf{C}$. Let $\mathcal{L}$ be the set consisting of all $n \times n$ matrices T that satisfy

$$T^{tr} A + AT = 0.$$

(Here T^{tr} is the transpose of T.) Let $[T_1, T_2] = T_1 T_2 - T_2 T_1$. Prove that $\mathcal{L}$, with the Lie bracket $[\ ,\ ]$, is a Lie algebra.

(ii) Let T be a matrix satisfying the above condition. Assume that T is nilpotent, i.e., $T^m = 0$ for some m. Let

$$\exp(T) = \sum_{k=0}^{m-1} \frac{T^k}{k!} \,.$$

Prove that

$$(\exp T)^{tr} A \exp(T) = A.$$

(iii) Let $n = 2l$ and

$$A = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix},$$

where I_l is the identity matrix of size l. Show that $T \in \mathcal{L}$ if and only if

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{22} = -T_{11}^{tr}$, and T_{12}, T_{21} are symmetric $l \times l$ matrices.

Let H be the set of diagonal matrices in $\mathcal{L}$. Assuming that H is a Cartan subalgebra of $\mathcal{L}$ find the Cartan decomposition of $\mathcal{L}$:

$$\mathcal{L} = H \oplus \bigoplus_r \mathbf{C} e_r.$$

(You must define the corresponding elements e_r).

For $h = \operatorname{diag}(\lambda_1, \ldots, \lambda_l, -\lambda_1, \ldots, -\lambda_l) \in H$, evaluate $[h, e_r]$ explicitly.

END OF PAPER