UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 9 June, 2005 1.30 to 3.30

PAPER 39

POISSON PROCESSES

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain carefully what is meant by a Poisson process Π with mean measure μ on a space S. State without proof sufficient conditions for such a process to exist.

The points of Π are coloured randomly either red or green, the probability of any point being red being $r(0 < r < 1)$ and the colours of different points being independent. Show (without appeal to any general theorem about Poisson processes) that the red and the green points form independent Poisson processes.

2 Show that, if $Y_1 < Y_2 < Y_3 < \ldots$ are the points of a Poisson process on $(0, \infty)$ with constant density λ, then

$$\lim_{n\to\infty} Y_n/n = \lambda$$

with probability one.

A Poisson process Π on $(0, 1)$ has density

$$\Lambda(x) = x^{-2}(1-x)^{-1}\,.$$

Show that the points of Π can be labelled as

$$\ldots < X_{-2} < X_{-1} < \tfrac{1}{2} < X_0 < X_1 < \ldots$$

and that

$$\lim_{n\to-\infty} X_n = 0\,,\ \lim_{n\to\infty} X_n = 1\,.$$

Prove that

$$\lim_{n\to-\infty} (-n)X_n = 1$$

with probability one. What can you say about X_n as $n \to +\infty$?

3 A model of a rainstorm falling on a level surface (taken to be the plane $\mathbb{R}^2$) describes each raindrop by a triple (X, T, V), where $X \in \mathbb{R}^2$ is the horizontal position of the centre of the drop, T is the instant at which the drop hits the plane, and V is the volume of water in the drop. The points (X, T, V) are assumed to form a Poisson process on $\mathbb{R}^4$ with a given density $\lambda(x, t, v)$. The drop forms a wet circular patch on the surface, with centre X and a radius that increases with time, the radius at time $(T + t)$ being a given function $r(t, V)$. Find the probability that a point $\xi \in \mathbb{R}^2$ is dry at time τ, and show that the total rainfall in the storm has expectation

$$\int_{\mathbb{R}^4} v\lambda(x, t, v)dx\ dt\ dv$$

if this integral converges.

(Any general theorems used must be carefully stated, but should not be proved.)

END OF PAPER