UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Thursday 9 June, 2005 9 to 12

PAPER 37

QUANTUM INFORMATION THEORY

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(a)** Consider a composite system AB which is in a pure state $|\Psi_{AB}\rangle$. What is the joint entropy $S(A, B)$ of the system? Define the conditional entropy $S(B|A)$ and show that it is negative if and only if the state $|\Psi_{AB}\rangle$ is entangled.

(b) Prove that any quantum state can be *purified*. Use purification to prove the triangle inequality

$$S(A, B) \geq |S(A) - S(B)|.$$

Here $S(A)$ and $S(B)$ denote the von Neumann entropies of the subsystems A and B respectively.

2 **(a)** Derive the Schmidt decomposition of a pure state $|\Psi_{AB}\rangle$ of a composite system AB. Use it to prove that the density matrices of the subsystems A and B have the *same* non-zero eigenvalues.

(b) Find the Schmidt numbers for the following states:

$$(i) \quad |\Phi\rangle = \frac{1}{\sqrt{3}} \left[|10\rangle - |01\rangle + |11\rangle \right]$$

$$(ii) \quad |\Psi\rangle = \frac{1}{2} \left[|00\rangle - |01\rangle - |10\rangle + |11\rangle \right]$$

3 **(a)** Can the Bell state

$$|\Psi^{-}\rangle := \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right)$$

be transformed to the bipartite pure state

$$|\Phi\rangle = \cos\phi |01\rangle + \sin\phi |10 >,$$

where $0 \leq \phi \leq \pi/4$, by local operations and classical communications (LOCC) alone ? Justify your answer.

(b) Prove that the Schmidt number of a pure state cannot be increased by LOCC alone.

4 **(a)** What is a discrete memoryless classical channel? Define its capacity. Consider a channel with input and output alphabet $I = \{1, 2, 3\}$. With probability 2/3 any input letter remains unaffected, while with probability 1/3 it gets changed to the next letter. For example, if 3 is the input letter, then the output is 3 with probability 2/3 and 1 with probability 1/3. Find the capacity of this channel.

(b) The depolarizing channel is defined as

$$\Phi(\rho) = (1-p)I + \frac{p}{3}\sum_{k=1}^{3} \sigma_k \rho \sigma_k,$$

where I is the identity operator and σ_1, σ_2 and σ_3 denote the Pauli matrices σ_x, σ_y and σ_z respectively. Derive the effect of this channel on the Bloch sphere.

5 **(a)** Alice has two classical bits that she wants to send to Bob. However, she only has a quantum channel at her disposal, which she is allowed to use only once. Under what condition can she achieve her goal ? What is the protocol that she would use?

(b) The quantum [[7, 1, 3]] Steane code $\mathcal{X}_{\text{Steane}}$ has basis states

$$|\psi_{\text{ev}}\rangle = \frac{1}{\sqrt{8}} \sum_{\mathbf{x}\in\mathcal{C}_{\text{ev}}} |\mathbf{x}\rangle\,; \quad |\psi_{\text{odd}}\rangle = \frac{1}{\sqrt{8}} \sum_{\mathbf{x}\in\mathcal{C}_{\text{odd}}} |\mathbf{x}\rangle.$$

Here $\mathcal{C}_{\text{ev}}$ and $\mathcal{C}_{\text{odd}}$ denote the two sets of 8 codewords of the classical $(7, 4, 3)$ Hamming code $\mathcal{C}_H$, corresponding to codewords of even and odd weight, respectively. Prove that this code can correct, *non-degenerately*, a single *phase flip* error. [*Hint*: $\mathcal{C}_{\text{ev}}^{\perp} = \mathcal{C}_H$, where $\mathcal{C}_{\text{ev}}^{\perp}$ is the dual of the code $\mathcal{C}_{\text{ev}}$.]

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