UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 1.30 to 4.30

PAPER 31

CLASS FIELD THEORY

Attempt ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Write an essay on the decomposition group and the inertia group. You should explain how the orders of these groups are related to the factorisation of prime ideals.

(ii) Define the Artin map for an abelian extension of number fields L/K. Explain with proof what happens when L/K is replaced by a subextension E/K or a translation LE/E.

2 (i) State and prove a version of Hensel's lemma. Compute the group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ for p a rational prime.

(ii) Give a brief summary of the properties of the Hilbert norm residue symbol. Compute the following subgroups of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ and verify that they are exact annihilators with respect to the Hilbert norm residue symbol.

$$\Delta_1 = \{x \in \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 |\ \mathrm{ord}_p(x) \equiv 0 \pmod 2\}$$

$$\Delta_2 = \{x \in \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2 | \mathbb{Q}_p(\sqrt{x})/\mathbb{Q}_p \text{ is unramified } \}$$

3 Write an essay on the Herbrand quotient and its application to norm index computations. Give full details either in the case of local fields or in the case of number fields. You should end by outlining the proof of the Hasse norm theorem.

4 Write an essay explaining the relationship between the Artin reciprocity law and the computation of the Brauer group of a number field.

5 (i) State the classification theorem of class field theory. Quoting any results you need about conductors, deduce the existence of the Hilbert class field.

(ii) Find the Hilbert class field of $K = \mathbb{Q}(\sqrt{-30})$. Deduce that if p is a rational prime, then $p = 2x^2 + 15y^2$ is soluble for integers x, y if and only if $p = 2$ or $p \equiv 17, 23, 47, 113 \pmod{120}$.

END OF PAPER