UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 14 June, 2005 9 to 12

PAPER 30

CYCLOTOMIC FIELDS

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Notation: Throughout, p will denote an odd prime, and $\mathbb{Z}_p$ the ring of p-adic integers. For any ring A, A^* will denote the multiplicative group of units of A. Let $R = \mathbb{Z}_p[[T]]$ be the ring of formal power series in an indeterminate T with coefficients in $\mathbb{Z}_p$. Finally, $\phi : R \to R$ will be the ring homomorphism defined by $\phi(f)(T) = f((1+T)^p - 1)$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove that there exists a unique $\mathbb{Z}_p$ linear map $S : R \to R$ such that, for all f in R,

$$(\phi \circ S)(f) = \frac{1}{p} \sum_{\zeta \in \mu_p} f(\zeta(1+T) - 1)\,,$$

where μ_p denotes the group of p-th roots of unity. Prove that S is a left inverse of ϕ, i.e. $S \circ \phi$ is the identity map of R.

2 Define the Coleman norm map $N : R^* \to R^*$. If $f(T)$ in R satisfies

$$\phi(f)(T) \equiv 1 \bmod p^k R$$

for some integer $k \geqslant 1$, prove that $f(T) \equiv 1 \bmod p^k R$. Hence show that if f is any element of R^* with $N(f) = f$ and $f \equiv 1 \bmod pR$, then necessarily $f = 1$. Let A denote the ring of formal power series in T with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and let W be the subgroup of R^* consisting of all units f such that $Nf = f$. Deduce that A is naturally isomorphic to the direct product of W and A^*.

3 Define the Iwasawa algebra $\Lambda(G)$ of any profinite abelian group G, and explain, without detailed proof, its measure-theoretic interpretation.

State Mahler's theorem on continuous functions on $\mathbb{Z}_p$, and deduce from it a canonical bijection

$$\lambda : R \to \Lambda(\mathbb{Z}_p)\,.$$

If f is any element of R, prove that, for all integers $k \geqslant 0$,

$$\int_{\mathbb{Z}_p} x^k d\lambda(f) = (D^k f)(0)$$

where $D = (1+T)\frac{d}{dT}$.

4 Write an essay outlining the proof of Iwasawa's theorem, explaining briefly how it led him to formulate the main conjecture on cyclotomic fields.

END OF PAPER