UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 10 June, 2005 1.30 to 4.30

PAPER 3

REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

*Attempt **FOUR** questions.*

*There are **EIGHT** questions in total.*

The questions carry equal weight.

Throughout, for $n \in \mathbb{N}$, $G = \Sigma_n$ is the symmetric group of degree n and F is a field of characteristic p.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove the Submodule Theorem, which states that every submodule of the permutation module, M^{λ}, of Σ_n acting on the Young subgroup Σ_{λ} corresponding to the partition λ of n, either contains the Specht module S^{λ} or is contained in $S^{\lambda\perp}$. In the course of your proof all terms which were introduced during the lecture course should be defined. Deduce that $D^{\lambda} = S^{\lambda}/(S^{\lambda} \cap S^{\lambda\perp})$ is zero or absolutely irreducible. State necessary and sufficient conditions, in terms of λ and the characteristic of F, for D^{λ} to be non-zero, and outline a justification for the conditions.

2 What does it mean to say that λ dominates μ (written $\lambda \trianglerighteq \mu$) for partitions λ and μ of n? Consider the permutation module M^{μ} of G acting on the Young subgroup Σ_{μ} corresponding to the partition μ of n. Show that every composition factor of M^{μ} has the form D^{λ} (as defined in Question 1), with $\lambda \triangleright \mu$ except if μ is p-regular when D^{μ} occurs precisely once. State Young's Rule which determines these factors exactly over $\mathbb{Q}$. Illustrate Young's Rule by using it to calculate $S^{(3,2)} \otimes S^{(2)} \uparrow^{\Sigma_7}$ over $\mathbb{Q}$.

3 Describe, with justification, the general form of the composition factors of the Specht module S^{μ} over F. Hence give the general form of the matrix (M) recording the composition factors of the Specht modules over F with rows and columns suitably ordered (the results of the previous question may be quoted if desired). Why is (M) the p-modular decomposition matrix for G?

Construct (M) explicitly when $n = 3$ for any characteristic.

4 State the Littlewood-Richardson Rule, defining all terms used. Illustrate the Rule by calculating the Littlewood-Richardson coefficient of $S^{(5,3,2,1)}$ in $S^{(3,2,1)} \otimes S^{(3,2)}$. Deduce the Determinantal Form $[\lambda] = \det([\lambda_i + j - i])$. Illustrate the Form by computing $[3, 2, 2]$.

5 State and prove the Branching Theorem. State also the Hook Rule on the dimensions of the ordinary irreducible representations of G.

Show that, for each n, $S^{(n)}$ and $S^{(1^n)}$ are the only one-dimensional ordinary representations of G. Show also that for $2 \leqslant n \neq 4$, the lowest dimension $\neq 1$ of an ordinary irreducible representation of G is $n - 1$, and that for $2 \leqslant n \neq 6$, $S^{(n-1,1)}$ and $S^{(2,1^{n-2})}$ are the only ordinary irreducible representations of G which are of dimension $n - 1$. Find the exceptional cases for $n = 4$ and $n = 6$.

6 State the Kernel Intersection Theorem, ensuring that you define carefully all the terms you use. State also the semistandard homomorphism theorem.

Prove that the Specht module S^μ contains a non-zero vector which is fixed under the G-action if and only if, for every i, $\mu_i \equiv -1 \pmod{p^{t_i}}$, where t_i is the least non-negative integer such that $\mu_{i+1} < p^{t_i}$.

For which p-regular partition μ is the unique top composition factor, D^μ of S^μ isomorphic to the alternating representation $S^{(1^n)}$? You may assume, without proof, that $S^\lambda \otimes S^{(1^n)}$ is isomorphic to the dual of $S^{\lambda'}$.

7 State the Murnaghan-Nakayama Rule. Illustrate it by computing the value of $\chi^{(3^3)}$ on the element (1234)(56)(789).

Recall that a hook diagram is one of the form $[x, 1^y]$. Use the Littlewood-Richardson Rule to show that, unless both $[\alpha]$ and $[\beta]$ are hook diagrams, $[\alpha][\beta]$ contains no hook diagrams. Deduce that if $[\alpha] = [a, 1^{n-r-a}]$ and $[\beta] = [b, 1^{r-b}]$ then

$$[\alpha][\beta] = [a+b, 1^{n-a-b}] + [a+b-1, 1^{n-a-b+1}] + \text{some non-hook diagrams}\,.$$

Finally deduce a special case of the Rule as follows. Suppose that ρ is an n-cycle, and ν is a partition of n. Show that $\chi^\nu(\rho) = (-1)^{n-x}$ if $[\nu]$ is the hook diagram $[x, 1^{n-x}]$, otherwise it is zero.

8 Explain the abacus notation for a partition. Define β-numbers and describe their relationship to first column hook lengths. Use this notation to show that the p-core of a partition is well-defined.

What is a p-block of G? State the "Nakayama Conjecture" on the p-block structure of G and write an essay indicating the main steps in its proof.

END OF PAPER