UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 11

PAPER 29

DIOPHANTINE ANALYSIS AND TRANSCENDENCE THEORY

Attempt ***TWO*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove that π is transcendental. State the more general Lindemann theorem and show that it implies that $\cos\alpha\cos\beta$ is transcendental for all non-zero algebraic α, β.

2 State and prove Siegel's lemma on the solution of linear equations.

Define Siegel's E-functions. State the Siegel-Shidlovsky theorem and specify the particular E-functions that give Lindemann's theorem.

3 Using Mahler's method, show that the Fredholm series $\sum_{n=0}^{\infty} z^{l^n}$, where l is an integer $\geqslant 2$, assumes transcendental values for all algebraic z with $0 < |z| < 1$.

Prove that, for $l \geqslant 3$ and $z = \frac{1}{2}$, the transcendence of the Fredholm series can be obtained directly from the Thue-Siegel-Roth theorem.

4 State the Gelfond-Schneider theorem and outline a proof.

Show that the theorem implies that if α, β are algebraic numbers, not 0 or 1, then $\log\alpha/\log\beta$ is either rational or transcendental. State a generalisation applicable to arbitrarily many logarithms.

5 Assuming an appropriate estimate for a logarithmic form, show that the equation $\alpha x + \beta y = 1$, where α, β are non-zero elements of an algebraic number field K, has only finitely many solutions in units x, y in K. Indicate how this leads to a proof of Thue's theorem on the equation $F(x, y) = m$.

6 State the *abc*-conjecture. Show that it implies that the Fermat equation

$$x^n + y^n = z^n$$

has only finitely many solutions in positive integers x, y, z and $n > 3$.

Show further that the conjecture implies that the exponential equation

$$ax^n - by^n = c\,,$$

where a, b, c are given positive integers, has only finitely many solutions in positive integers x, y and $n > 2$. Indicate how the latter can be established from the theory of linear forms in logarithms without any unproved hypothesis.

END OF PAPER