UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 9 to 12

PAPER 28

PRIME NUMBERS

Attempt ***ONE*** *question from each of Sections A, B and C.*

There are ***SEVEN*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 State and prove a version of the Poisson summation formula (you may assume any results on the uniqueness of Fourier series that you need).

Define the completed ζ-function $\Xi(s)$. Show that it is meromorphic on $\mathbf{C}$, and that it is analytic except for simple poles at $s = 0$ and $s = 1$.

2 (i) Define the super-completed ζ-function $\xi(s)$. Show that it is an integral function of order one (you may assume any facts about the Γ-function that you need). Quoting any results from the theory of integral functions of order one that you need, deduce the partial fraction expansion for ζ'/ζ.

(ii) State and prove Perron's formula. Sketch, very briefly, how this may be used together with the partial fraction expansion obtained in (i) to link the distribution of primes with the zeros of ζ.

Section B

3 (i) Prove that there is an absolute constant $c > 0$ such that any zero $\rho = \beta + i\gamma$ of the ζ-function in the critical strip satisfies

$$\beta < 1 - \frac{c}{\log(|\gamma| + 2)} .$$

You may assume any bounds on Γ'/Γ that you need.

(ii) By using the explicit formula

$$\psi(x) = x - \sum_{|\rho|<T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right) ,$$

prove the prime number theorem in the form $\pi(x) \sim x/\log x$ (you may assume a bound on the number of zeros ρ with $|\rho| < T$).

(iii) Give a brief discussion of what bound for $|\psi(x) - x|$ could be obtained assuming a zero-free region of the form $\beta < 1 - c$, for some $c \in (0, 1/2)$.

4 State the partial fraction expansions for ζ'/ζ and for L'/L. State and prove Landau's theorem. What is meant by a Siegel zero? Show that at most one of the real, primitive, Dirichlet characters to modulus q has a Siegel zero. Show also that if $q_1 < q_2 < q_3 < \dots$ is the sequence of moduli to which there is a real, primitive, Dirichlet character with a Siegel zero then $q_{j+1} > q_j^2$ for all j.

Section C

5 State the large sieve inequality. Consider a set $A \subset \{1, \ldots, N\}$ with the following property. For each prime $p \leqslant X$, let $h_1, \ldots, h_{k(p)}$ be distinct residues modulo p, and suppose that if $a \in A$ then $a \not\equiv h_i (\mathrm{mod}\ p)$ for all $i = 1, \ldots, k(p)$. Prove that

$$|A| \leqslant (8N + X^2) \left(\sum_{p \leqslant X} \frac{k(p)}{p} \right)^{-1} .$$

Prove that the number of primes $p \leqslant N$ for which the least quadratic nonresidue $n(p)$ satisfies $n(p) > p^{0.01}$ is at most $C \log N$, for some absolute constant C.

6 Write an essay on the Selberg sieve as applied to the problem of estimating the number of twin primes less than N.

7 (i) Define the von Mangoldt function Λ , and state the Siegel-Walfisz theorem. Show that we have the asymptotic

$$\sum_{n \leqslant N} \Lambda(n) e(2n/5) \sim cN$$

for some constant c, which you should give explicitly.

(ii) Prove that if $a, q \geqslant 1$ are integers then

$$\left| \sqrt{2} - \frac{a}{q} \right| > \frac{1}{4q^2} .$$

Give a discussion of how one could prove that

$$\sum_{n \leqslant N} \Lambda(n) e(n\sqrt{2}) = O(N^{9/10}) .$$

You should indicate the basic structure of the argument but need not give technical details.

END OF PAPER