UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 9 to 11

PAPER 25

MORSE THEORY

*Attempt **TWO** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $SO(n) = \{X \in M_n(\mathbf{R}) | X^t X = I, \det X = 1\}$ be the special orthogonal group. [*You may assume that* $SO(n)$ *is a smooth submanifold of* $M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ *and that the embedded tangent space at* $X \in SO(n)$ *is given by the affine subspace*

$$\{X + AX | A^t = -A\}.]$$

Suppose $1 < c_1 < c_2 < \ldots < c_n$ and let C be the diagonal matrix with $c_1, c_2, \ldots, c_n$ on the diagonal. Define a function

$$f : SO(n) \to \mathbf{R} : X \mapsto \mathrm{tr}(CX).$$

Find the critical points of f and their indices. Deduce that $\chi(SO(3)) = 0$. [Hint: it may help to consider curves through $X \in SO(n)$ of the form

$$\theta \mapsto R^{ij}(\theta)X \qquad \text{and} \qquad \theta \mapsto XR^{ij}(\theta)$$

where $R^{ij}(\theta)$ is the special orthogonal matrix with entries

$$\left(R^{ij}(\theta)\right)_{kl} = \begin{cases} 1 & i \neq k = l \neq j \\ \cos\theta & i = k = l \text{ or } j = k = l \\ -\sin\theta & i = k, j = l \\ \sin\theta & i = l, j = k \\ 0 & \text{otherwise} \end{cases}$$

and $1 \leqslant i < j \leqslant n$.]

2 Suppose $f : M \to \mathbf{R}$ is a Morse function. For $0 \leqslant i \leqslant m = \dim(M)$ let $b_i = \operatorname{rank} H_i(M; \mathbf{Z})$ be the ith Betti number of M and c_i the number of index i critical points of f. Explain (without quoting the Morse inequalities) why $b_i \leqslant c_i$ for $0 \leqslant i \leqslant m$. State the Morse inequalities for f, and prove that they imply the above inequalities. Are they equivalent? Write down the homology $H_*(M; \mathbf{Z})$ of M under the assumption that the Morse inequalities are all equalities.

Let Σ_g be a closed smooth oriented real surface of genus $g \geqslant 1$. Find a lower bound for the number of critical points of a Morse function on $\Sigma_g \times \Sigma_g$. Describe a Morse function on $\Sigma_g \times \Sigma_g$ with this number of critical points. Is this the minimum number of critical points of any smooth function on $\Sigma_g \times \Sigma_g$? Justify your answer.

3 Explain what is meant by a $(\lambda, m+1-\lambda)$-surgery on an m-manifold M.

Suppose $\imath : S^1 \times D^1 \hookrightarrow T^2$ is an embedding such that $\imath(S^1 \times \{0\})$ is a (p,q)-curve on the torus T^2 i.e. it represents the homology class $pa + qb$ where $a, b \in H_1(T^2; \mathbf{Z})$ are the two standard generators. Let M be the manifold obtained from T^2 by surgery with respect to this embedding. Compute $H_*(M; \mathbf{Z})$ in the case $p, q > 0$. Deduce a necessary and sufficient condition on the strictly positive integers p and q for a (p,q)-curve to exist. [*You may assume that a closed, oriented smooth real surface is diffeomorphic to a surface of genus g for some $g \geqslant 0$.*]

4 Give an explicit Morse function on the real projective space $\mathbf{RP}^m$. Find, with proof, the critical points and indices. Explain carefully how to construct the Morse–Smale complex of your Morse function. Compute $H_*(\mathbf{RP}^m; \mathbf{Z})$.

5 State the h-cobordism theorem. Which of the conditions are *necessary* for the conclusion to hold? Give proofs or counterexamples as appropriate.

Prove that (i) a cobordism which possesses a Morse function with an odd number of critical points is not trivial, and (ii) a cobordism which possesses a Morse function with two critical points p and q, of respective indices λ and μ where $\lambda \leqslant \mu$, can only be trivial if $\mu = \lambda + 1$ and the intersection number $S_U(p) \cdot S_L(q)$ of the upper and lower spheres, defined with respect to an appropriate gradient-like vector field, is ± 1.

Construct a non-trivial cobordism of S^3 to itself. [Hint: consider $(2,2)$-surgery along a knot $S^1 \hookrightarrow S^3$.]

END OF PAPER