UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 1.30 to 4.30

PAPER 24

RATIONAL HOMOTOPY THEORY

You must answer ***QUESTION 1****,*

and ***ANY THREE*** *questions from 2 - 6.*

Question 1 is worth 34 points;

Every other question is worth 22 points,

for a maximum total of 100 points.

No more than ***THREE*** *of your marks on Questions 2 - 6*

will be taken into account.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on the rationalisation $X_{\mathbf{Q}}$ of a topological space X, describing its construction from the Postnikov tower. You should define the key concepts, such as Eilenberg-MacLane spaces, principal fibrations, k-invariants, and you should state (without proof) the key theorems that you use. You should also discuss the significance of $X_{\mathbf{Q}}$ in relation to maps from X to rational spaces.

[*Make any assumptions on $\pi_1 X$ that you find convenient.*]

2 (i) Define the Whitehead bracket and list its basic algebraic properties. For bilinearity (but not for the other properties), also discuss the exceptional case of π_1.

(ii) The *Hopf invariant* of a map $f : \mathbf{S}^{4n-1} \to \mathbf{S}^{2n}$ is the square of the generator of H^{2n} in the space X_f, obtained by attaching $\mathbf{D}^{4n}$ to $\mathbf{S}^{2n}$ via f. Show that the Whitehead square of the generator $\alpha \in \pi_{2n}(\mathbf{S}^{2n})$ has Hopf invariant 2.

3 Prove that the rational homotopy groups of a simply connected space Y form a free Lie algebra under the Whitehead bracket iff Y is rationally equivalent to the suspension of some space X. Show, in that case, that the homology of ΩY, with the Pontryagin product, is the free tensor algebra generated by the reduced homology $\tilde{H}_*(X;\mathbf{Q})$.

[*Any general theorems that you use must be clearly stated.*]

4 By computing minimal models, prove that the space $\mathbf{CP}^5/\mathbf{CP}^2$ (obtained from $\mathbf{CP}^5$ by collapsing $\mathbf{CP}^2$ to a point) has the rational homotopy type of $\mathbf{S}^6 \vee \mathbf{S}^8 \vee \mathbf{S}^{10}$.

5 Let X be simply connected and assume that the Hurewicz homomorphism is surjective. Show that X is formal, and is rationally equivalent to a wedge of spheres.

[*Hint: You may want to show that the space is formal and that the cup-product on reduced cohomology vanishes. Consider for that the projection $A^+ \to A^+/A^+ \cdot A^+$, in a minimal model A^*.*]

6 Consider the degree 1 map $f : \mathbf{CP}^n \to \mathbf{S}^{2n}$.

(i) Describe the induced map f^* on minimal DGA models of these spaces.

(ii) Determine the minimal model for the homotopy fibre of this map.

[*You must explain your reasoning. If you have trouble, try the case $n = 2$. Find the rational homotopy groups in general.*]

END OF PAPER