UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 10 June, 2005 9 to 12

PAPER 22

STABLE HOMOTOPY THEORY

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define: (i) spectrum, (ii) map of spectra, (iii) Ω-spectrum, (iv) cell spectrum.

(b) For an Ω-spectrum E, define E_* and E^*, the (reduced) homology and cohomology theories on based spaces. (You do not have to give the definition of homology or cohomology theory or prove that your answer specifies one.)

(c) Let D and E be cell Ω-spectra, what is the relationship between:

(i) The set of maps of spectra from D to E.

(ii) The set of maps in the stable category from D to E.

(iii) The set of maps of cohomology theories on based spaces from D^* to E^*.

Please be as specific as you can. (You do not have to provide proofs.)

2 Let U be a universe, X a compact unbased space, and $f\colon X \to \mathcal{L}(U)$ a map from X to the space of linear isometries from U to $\mathbb{R}^\infty$. Let $\mathcal{I}$ be an indexing set for U, and let $\mathcal{F}$ be a flag subordinate to X.

(a) Write the definition of the point-set functor $X \ltimes (-)$ from $\mathcal{I}$-spectra to spectra (for this data).

(b) Given $g\colon Y \to \mathcal{L}(\mathbb{R}^\infty)$, let $h\colon Y \times X \to \mathcal{L}(U)$ be the map that sends $(y, x) \in Y \times X$ to $g_y \circ f_x\colon U \to \mathbb{R}^\infty$. Show that there is a natural isomorphism in the stable category $Y \ltimes (X \ltimes (-)) \cong (Y \times X) \ltimes (-)$.

3 (a) Define ring spectrum and commutative ring spectrum.

(b) Let X be a space and E a ring spectrum. Show that $F(X_+, E)$ is a ring spectrum.

4 Let E be a connective spectrum with $\pi_0 E = \mathbb{Z}$.

(a) Show that if X is an $(n-1)$-connected based space and $n \geqslant 2$, then $E_n X \cong \pi_n X$.

(b) Show that if $f\colon Y \to Z$ is a map of simply connected spaces that induces an isomorphism on E-homology, then f is a weak equivalence.

[You may assume the Whitehead theorem for ordinary homology.]

5 Let D be a spectrum that is not a wedge summand (in the stable category) of a wedge of finite spectra. Show that there is a spectrum E and a *non-trivial* map in the stable category $D \to E$ that induces the zero map of the associated homology theories. (Hint: Construct E as a cofiber $X \to D$ with X a wedge of finite spectra.)

END OF PAPER