UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 9 June, 2005 9 to 12

PAPER 21

K3 SURFACES

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Give the definition of the scroll $\mathbb{F} = \mathbb{F}(a_1, \ldots, a_n)$ and describe the standard line bundles L, M on $\mathbb{F}$.

Compute all intersection numbers of L, M and state the formula for the canonical class of $\mathbb{F}$.

Describe the natural embeddings of $\mathbb{F}$ in projective space and check that $\deg(\mathbb{F}) = \operatorname{codim}(\mathbb{F}) + 1$.

(b) Prove that

$$\mathbb{F}(a_1, \ldots, a_n) \cong \mathbb{F}(b_1, \ldots, b_n) \quad \text{if and only if} \quad \{a_1, \ldots, a_n\} = \{b_1 + c, \ldots, b_n + c\}$$

for some c.

2 Give the definition of a trigonal curve. Show that the canonical image of a trigonal curve C of genus g is contained in a surface scroll $\mathbb{F}(a_1, a_2) \subset \mathbb{P}^{g-1}$ where $g = a_1 + a_2 + 2$ and the canonical morphism $\mathbb{F}(a_1, a_2) \to \mathbb{P}^1$ induces the g^1_3. Suppose that $a_1 \leqslant a_2$, set $a = a_2 - a_1$. Let $L \subset \mathbb{F}_a$ and $B \subset \mathbb{F}_a$ be the fibre and the negative section. Verify that $\mathbb{F}(a_1, a_2)$ is $\mathbb{F}_a$ embedded by $a_2 L + B$ and show that

$$C \in |(a + a_2 + 2)L + 3B|$$

Show that a general element of this linear system is nonsingular if and only if $3a \leqslant g + 2$.

3 (a) Give definitions and examples of monogonal, hyperelliptic and trigonal linear systems on a K3 surface. Briefly explain why your examples are indeed examples.

(b) State the first, second and third dichotomy for linear systems on K3 surfaces.

4 (a) Give the definition of polarised Hodge structure. Give the definition of the Griffiths domains $\check{D}$ and D parameterising polarised Hodge structures.

(b) Discuss in detail the weight one case: Fix the nondegenerate antisymmetric form

$$\psi = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$$

on $\mathbb{Z}^{2r}$. Consider Hodge structures on $\mathbb{Z}^{2r}$ polarised by the form ψ and show that $\check{D} = SpGr(r, 2r)$ is the Grassmannian of r-dimensional complex subspaces $H \subset \mathbb{C}^{2r}$ which are isotropic for ψ. Calculate the dimension of $\check{D}$. Describe an explicit identification:

$$D = \{Z \in M_r(\mathbb{C}) \mid {}^tZ = Z, \, \Im Z > 0\}$$

and describe explicitly the action of $Sp_{2r}(\mathbb{Z})$ on D.

(c) Briefly describe the Griffiths domains parameterising weight two Hodge structures on $H^2(X, \mathbb{Z})$ where X is a K3 surface.

5 (a) Give the definition of variation of Hodge structure.

(b) Let $f \colon X \to S$ be a smooth family of Kähler manifolds. Briefly explain how the cohomology of fibres gives rise to a variation of Hodge structure. Sketch the proof of Griffiths transversality.

(c) Describe the lattice $L = H^2(X, \mathbb{Z})$ for a K3 surface X.

(d) Briefly state your favourite version of the Torelli theorem for K3 surfaces.

END OF PAPER