UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 9 to 12

PAPER 20

SPECTRAL GEOMETRY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

Where necessary, you may quote standard results from analysis without proof.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Given an account of the development of Spectral Geometry since Kac's paper of 1966.

2 Define the Hessian of a function on a Riemannian manifold. Defining the Laplacian to be the trace of the Hessian, derive an expression for the Laplacian of a function in terms of its values along suitable geodesics.

Assuming the existence of isospectral non-isometric 4-dimensional tori, prove the existence of isospectral non-isometric n-dimensional tori for all $n > 4$.

3 Define the heat kernel, heat trace and heat invariants of a Riemannian manifold. Assuming the existence and uniqueness of the heat kernel derive an expression for the heat trace involving the spectrum of the Laplacian.

Stating clearly, but without proof, any properties of the asympototic expansion of the heat kernel that you require, deduce that isospectral manifolds have the same dimension and volume.

4 State the relation between the heat kernels of two Riemannian manifolds M and N where N is a finite normal Riemannian covering of M with covering transformation group U. Derive Sunada's formula for the heat trace of M when U is a subgroup of a larger group T of isometries of N.

Prove that, if N is the universal covering of M_0 with $\pi_1(M_0) \cong T$ and if U_1 and U_2 are Gassman equivalent subgroups of T, then the quotient spaces $M_i = U_i \backslash N$ are isospectral provided the metrics are chosen so that all the coverings are local isometries.

Explain *briefly* how for any finitely presented group T we may construct such a manifold M_0 and how we may ensure that M_1 and M_2 are not isometric.

5 Given a finite group T, describe how to construct, from a suitable hyperbolic polygon $\mathcal{D}$, surfaces M and M_0 having Riemannian metrics with constant curvature -1 except possibly at cone-like singularities, such that M is a covering of M_0 with covering transformation group T.

Explaining any terms involved, state the formulae that determine the Euler characteristics of M and M_0 and the order of any potential cone singularities.

If U is a subgroup of T and $M_1 = U\backslash M$ state the necessary and sufficient condition for M to be a normal covering of M_1 and state the value of $\chi(M_1)$ in that case.

Describe the hyperbolic polyhedra $\mathcal{D}$ that, for certain groups T and subgroups U_i as well as suitably chosen generators A_k of T, will produce isospectral Riemann surfaces of genus 3 and 4 respectively.

[You should specify the required vertex angles of $\mathcal{D}$ as well as the orders $|T|, |U_i|$ and $|A_k|$; but you need not prove the existence of $\mathcal{D}$, nor of the groups and their generators.]

END OF PAPER