UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 1.30 to 4.30

PAPER 19

GEOMETRISATION OF 3-MANIFOLDS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

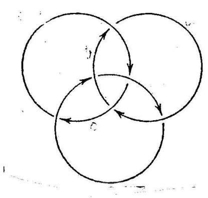
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let k be a knot in S^3. Obtain a presentation of the knot group $\pi_1(S^3 - k)$. Apply the same method to the 3-component link l (the Borromean rings) illustrated below:



Show that $\pi_1(S^3 - l)$ is generated by x_1, x_2, x_3 subject to the relations that x_1 commutes with the commutator $[x_2^{-1}, x_3]$ and x_2 with the commutator $[x_3^{-1}, x_1]$. Explain why the commutator quotient group is free-abelian of rank 3. By mapping x_3 to $\begin{pmatrix} i & 1 \\ 2i & 2-i \end{pmatrix}$, or otherwise, show that there is a faithful representation of $\pi_1(S^3 - l)$ in $PSL_2(\mathbb{C})$ and deduce that $S^3 - l$ has a hyperbolic structure.

[You may assume that the discrete subgroup $PSL_2(\mathbb{Z}[i])$ of $PSL_2(\mathbb{C})$ has a presentation $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,
$U = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ subject to the relations $TU = UT, L^2 = (TL)^2 = (UL)^2 = (AL)^2 = A^2 = (TA)^3 = (UAL)^3 = 1_2$.]

2 Let M^3 be a connected, compact, orientable 3-manifold without boundary. If $\pi_1(M^3) = \Gamma$ is infinite, torsion free and abelian, show that $\Gamma \cong \mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. Show further that, if Γ is infinite, torsion free and nilpotent, we must include groups with presentation

$$A \lhd \Gamma \twoheadrightarrow \mathbb{Z},$$

where the generator of Γ/A acts on $A \cong \mathbb{Z} \times \mathbb{Z}$ by means of the matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{Z}$.

If the centre $\zeta(\Gamma)$ has trivial intersection with the subgroup A show that Γ is abelian. Exhibit an example of a torsion free extension of such an abelian group by the finite cyclic group $\mathbb{Z}/4$. For which other finite groups Q do such torsion free extensions exist?

3 Let Γ be a PD^n-group. Explain how the duality between homology and cohomology follows from the condition

$$H^k(\Gamma, \mathbb{Z}\Gamma) \cong \begin{cases} \mathbb{Z}, & k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that if Γ_1 has finite index in Γ, then Γ is a PD^n-group if and only if Γ_1 is a PD^n-group.

Why would you expect a PD^2-group to be a surface group?

4 Outline the main steps in the proof of the Loop Theorem, and give two applications of it to the classification of 3-manifolds.

5 Write an essay on the isometrics of a C^∞-manifold, paying special attention to dimension 3.

END OF PAPER