UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 9 to 12

PAPER 18

FIBRE BUNDLES

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

As in the course, you may assume that whatever spaces come up are paracompact, and homotopy equivalent to CW complexes.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $SO(n)$ denote the group of linear automorphisms of $\mathbf{R}^n$ which preserve the standard metric and have determinant 1. Use fibre bundles to compute the homotopy groups $\pi_i SO(n)$ for $0 \leqslant i \leqslant 2$ and $n \geqslant 1$. Also, compute $\pi_3 SO(n)$ for $1 \leqslant n \leqslant 4$. [You may need to use the fact that $\pi_4 S^3 \cong \mathbf{Z}/2$.]

2 Show that the smooth 4-manifold $\mathbf{R}P^2 \times \mathbf{R}P^2$ cannot be immersed in $\mathbf{R}^5$.

3 Given two rank-2 complex vector bundles E and F on a space X, show that there is a formula for the second Chern class $c_2(E \otimes F)$ in terms of the Chern classes of E and F. Find the formula explicitly.

4 (i) Given a fibration $F \to E \to S^r$, prove that there is an exact sequence

$$\cdots \to H_i(F, \mathbf{Z}) \to H_i(E, \mathbf{Z}) \to H_{i-r}(F, \mathbf{Z}) \to H_{i-1}(F, \mathbf{Z}) \to \cdots .$$

You may assume that a fibration over a contractible space is trivial. (Hint: write E as a union of two open subsets, and use excision.)

(ii) Using (i), compute the integral homology groups of the loop space ΩS^r for $r \geqslant 3$. Deduce that there is a map $S^{r-1} \to \Omega S^r$ which induces an isomorphism on homology in degrees at most $2r - 3$.

5 (i) Using the conclusion of question 4(ii), plus the relative Hurewicz theorem, prove the Freudenthal suspension theorem: for $r \geqslant 3$ and $i \leqslant 2r - 4$, there is an isomorphism $\pi_i S^{r-1} \cong \pi_{i+1} S^r$. Using this isomorphism plus the fact that $\pi_4 S^3 \cong \mathbf{Z}/2$, compute $\pi_{n+1} S^n$ for all $n \geqslant 1$.

(ii) Find the subgroup H of $G = O(n)$ such that $\mathbf{R}P^{n-1} = G/H$. Using the fibration $G/H \to BH \to BG$, show that $H^*(BO(n-1), \mathbf{Q}) \cong H^*(BO(n), \mathbf{Q})$ for n odd. Explain where your argument fails for n even.

END OF PAPER