UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 9 to 12

PAPER 15

DIFFERENTIAL GEOMETRY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

All manifolds and related concepts should be assumed to be smooth.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State the defining properties of the exterior derivative d for the differential forms on a manifold M and show that d exists and is uniquely determined. Show also that $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, for any vector fields X, Y and any differential 1-form ω on M.

Define the de Rham cohomology groups and state Poincaré lemma. For any $n > 1$, show that there is an injective linear map $H^n(S^n) \to H^{n-1}(S^{n-1})$ between de Rham cohomology groups of spheres.

2 Define the concept of an embedded submanifold.

Let $f : M \to N$ be a smooth map between manifolds M and N. Define the regular values of f and prove that if $q \in N$ is a regular value of f then $f^{-1}(q)$ is an embedded submanifold of M of dimension $\dim M - \dim N$. [*You need not give a proof of the Inverse Mapping Theorem between open domains in a Euclidean space but should include an accurate statement.*]

State three different equivalent formulations of orientability for a manifold. Now let q be a regular value of $f : M \to N$ as above and assume that the manifold M is orientable. Deduce that the manifold $f^{-1}(q)$ is orientable too.

3 Show that an orthogonal structure on a real vector bundle E is equivalent to a choice of an inner product on E. Explain what is meant by an orthogonal local trivialization. Show that any real vector bundle admits an inner product.

Give two different ways to define the notion of orthogonal connection A on E, relative to an inner product h, and prove that these two definitions are equivalent.

[*Existence of a partition of unity on a manifold can be assumed without proof provided the result is accurately stated. You may assume without proof that every covariant derivative arises from a connection.*]

4 Let V be a Riemannian manifold and M an embedded submanifold of V. State the Gauss formula relating the Levi–Civita connections on V and on M. State the Weingarten formula defining the shape operator on M and prove that the shape operator is symmetric (self-adjoint).

State and prove the Gauss theorem for the curvature of the metric on M induced from V. Explain how the latter result implies Gauss' *Theorema Egregium.*

[*You may use without proof the formula* $R(X, Y) = D_{[X,Y]} - [D_X, D_Y]$ *for the curvature operator.*]

5 Let M be a compact oriented manifold endowed with a Riemannian metric g. Explain what is meant by the volume form of g showing that this volume form is well-defined. Define the Hodge $*$-operator. Identify, giving justification, all the complex numbers which occur as the eigenvalues of $*$.

Recall that the Laplace–Beltrami operator for the differential forms on M is defined by $\Delta = d\delta + \delta d$, where $\delta\eta = (-1)^{n(r+1)+1} * d * \eta$, for a form η of degree $r \geq 1$ ($n = \dim M$), and $\delta f = 0$ for a function f. Show that δ is the formal adjoint of d with respect to the L^2 inner product. State the Hodge decomposition theorem and give a necessary and sufficient condition on a differential form $\beta \in \Omega^k(M)$ so that the equation $\Delta\alpha = \beta$ has solutions $\alpha \in \Omega^k(M)$. Show that the set of all solutions α (if they exist) forms an affine space with the underlying vector space isomorphic to the de Rham cohomology $H^k(M)$.

[*Standard results on the integration of differential n-forms on an n-dimensional manifold can be used without proof provided that these are clearly stated.*]

END OF PAPER