UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 1.30 to 4.30

PAPER 14

ALGEBRAIC GEOMETRY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of sheaves of abelian groups on a space X. Assuming the result that when $\mathcal{F}_1$ is flabby, the maps on sections $\mathcal{F}_2(U) \to \mathcal{F}_3(U)$ are surjective for all open $U \subset X$, deduce that when **both** $\mathcal{F}_1$ and $\mathcal{F}_2$ are flabby, so too is $\mathcal{F}_3$. Indicate briefly the construction of sheaf cohomology groups via flabby resolutions, and deduce that any flabby sheaf $\mathcal{F}$ has trivial higher cohomology on X.

Let X now denote a smooth irreducible curve. Describe what is meant by the *divisor class group* $\mathrm{Cl}(X)$. Prove that $\mathrm{Cl}(X)$ is isomorphic to all of the following groups (where $\mathcal{K}_X^*$ denotes the constant muliplicative sheaf of non-zero rational functions and $\mathcal{O}_X^*$ the multiplicative sheaf of nowhere vanishing regular functions on X):

(a) The cokernel of the map $H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.

(b) $\mathrm{Pic}(X)$, the group of invertible $\mathcal{O}_X$-modules, modulo isomorphism.

(c) $H^1(X, \mathcal{O}_X^*)$.

2 Describe a construction for the structure sheaf of a (not necessarily irreducible) affine variety. Describe briefly what is meant by an (abstract) variety $(X, \mathcal{O}_X)$, and what it means to say that two varieties are *birationally equivalent*. Define the ring of rational functions $\mathrm{Rat}(X)$ on a variety X, and show that birationally equivalent varieties have isomorphic rings of rational functions.

Suppose now that X is an affine variety with coordinate ring A. Show that any open dense subset $U \subset X$ is a finite union of basic open subsets $D(f)$ with $f \in A$, where $D(f) = \{P \in X : f(P) \neq 0\}$, with at least one being dense in X. If S denotes the multiplicative subset of A consisting of elements which are not zero-divisors, prove that $\mathrm{Rat}(X) \cong S^{-1}A$ as k-algebras, and that there is a canonical injection $A \hookrightarrow \mathrm{Rat}(X)$. For any given point $P \in X$, show that the local ring $\mathcal{O}_{X,P}$ may be regarded as a subring of $\mathrm{Rat}(Y)$, for Y an appropriate closed subvariety of X (which you should identify).

3 Define what is meant by a *quasi-coherent* $\mathcal{O}_X$-module. In the case when X is affine and M is a $k[X]$-module, describe the construction of the associated $\mathcal{O}_X$-module $\tilde{M}$ (a detailed check of the sheaf conditions for $\tilde{M}$ is not required), and show that it is quasi-coherent. Prove that $\tilde{M}$ is *coherent* if and only if M is a finitely generated $k[X]$-module. Assuming the result that every quasi-coherent $\mathcal{O}_X$-module on an affine variety X is of the form $\tilde{M}$, for M the module of global sections, deduce that, if $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a morphism of quasi-coherent $\mathcal{O}_Y$-modules on an arbitrary variety Y, then $\mathrm{Ker}(\phi)$ is a quasi-coherent $\mathcal{O}_Y$-module.

Suppose now that $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of varieties, and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In the case when both X and Y are affine, prove that $f_*\mathcal{F}$ is a quasi coherent $\mathcal{O}_Y$-module. Suppose now that Y is affine but X is a general variety, and that $X = \bigcup U_i$ is a finite cover of X by open affines. By considering an appropriate morphism

$$\bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j} f_*(\mathcal{F}|_{U_i \cap U_j}),$$

or otherwise, deduce that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Does the same result continue to hold when both X and Y are arbitrary varieties?

4 Describe the construction of the sheaves $\mathcal{O}_{\mathbf{P}^n}(m)$ on $\mathbf{P}^n(k)$, where m denotes any integer and k is any algebraically closed field. Identify the group of global sections, and show that its dimension as a vector space over k is $\binom{m+n}{n}$. If $i : \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$ denotes the inclusion of $\mathbf{P}^{n-1}$ as a hyperplane, explain briefly why the groups $H^r(\mathbf{P}^{n-1}, \mathcal{F})$ and $H^r(\mathbf{P}^n, i_*\mathcal{F})$ are isomorphic, for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^{n-1}$.

We denote by $h^i(\mathcal{O}_{\mathbf{P}^n}(m))$, for $i \geq 0$, the dimension of $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m))$ as a vector space over k. Suppose now we are given the fact that, for r any dimension, $h^i(\mathcal{O}_{\mathbf{P}^r}(m)) = 0$ for all $i > 0$ and for all m sufficiently large. Using only general results on cohomology, prove by induction that $h^i(\mathcal{O}_{\mathbf{P}^n}(m)) = 0$ for all integers m and for all $0 < i \neq n$, and that $h^n(\mathcal{O}_{\mathbf{P}^n}(m)) = \binom{-m-1}{n}$ for all m.

Assuming the fact that the canonical line bundle on $\mathbf{A}^n$ is free of rank one generated by $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$, calculate the canonical line bundle for $\mathbf{P}^n$, and check that the above formulae are consistent with the statement of Serre Duality.

END OF PAPER