UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 9 to 12

PAPER 12

PROBABILISTIC COMBINATORICS

Attempt ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

In all questions you may assume without proof any form of the Chernoff bounds, including the following. If X is a sum of n independent indicator random variables with $\mathbb{E}(X) = \mu$ and $0 < c < 1$, then

$$\mathbb{P}(|X - \mu| \geq c\mu) \leq 2e^{-c^2\mu/3}.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $E_1, \dots, E_n$ be events. State and prove the Jordan-Bonferroni inequalities for the probability that exactly k of the events E_i hold.

Deduce that if X is a random variable taking non-negative integer values, and $\mathbb{E}_r(X) r^k/r! \to 0$ as $r \to \infty$, then

$$\mathbb{P}(X = k) = \frac{1}{k!} \sum_{r=0}^{\infty} (-1)^r \mathbb{E}_{k+r}(X)/r!.$$

For each $r \geq 1$, give an example of a random variable X taking non-negative integer values such that $\mathbb{E}_r(X)$ is finite but $\mathbb{E}_{r+1}(X)$ is infinite.

2 Let H be a fixed strictly balanced graph with v vertices, e edges, and a automorphisms. Writing $X_H(G)$ for the number of subgraphs of G isomorphic to H, show that if $p = cn^{-v/e}$ with c constant, then $X_H(G(n,p)) \xrightarrow{\mathrm{d}} \mathrm{Po}(c^e/a)$. (You may assume that if $\lim_{n\to\infty} \mathbb{E}_r(X_n) = \lambda^r$ for all $r \geq 1$, then $X_n \xrightarrow{\mathrm{d}} \mathrm{Po}(\lambda)$.)

Let K_4^+ be the five-vertex graph obtained by adding an edge to K_4. Is K_4^+ strictly balanced? Show that if $p = cn^{-2/3}$ with c constant, then

$$\lim_{n\to\infty} \mathbb{P}\left(k - \varepsilon < \frac{X_{K_4^+}(G(n,p))}{4cn^{1/3}} < k + \varepsilon \right) = \frac{(c^6/24)^k}{k!} e^{-c^6/24}$$

for any integer $k \geq 0$ and any $0 < \varepsilon < 1$.

3 For $k \geq 2$ fixed, let $G'_{k-\mathrm{out}}$ be the random directed graph on $[n] = \{1, 2, \dots, n\}$, $n > k$, obtained as follows: for each vertex i, choose a set S_i of k vertices from $[n] \setminus \{i\}$, with each of the $\binom{n-1}{k}$ possible choices equally likely. For different i, the S_i are taken to be independent. In $G'_{k-\mathrm{out}}$ there is an edge from i to j if and only if $j \in S_i$. Let $G_{k-\mathrm{out}}$ be the simple (undirected) graph underlying $G'_{k-\mathrm{out}}$, so there is an edge ij in $G_{k-\mathrm{out}}$ if and only if $j \in S_i$, $i \in S_j$, or both.

(a) Show that

$$\mathbb{P}(G_{2-\mathrm{out}} \text{ is connected}) \to 1 \text{ as } n \to \infty.$$

(b) Deduce that for any fixed $k \geq 2$,

$$\mathbb{P}(G_{k-\mathrm{out}} \text{ is connected}) \to 1 \text{ as } n \to \infty.$$

4 When is a sequence $X_0, \ldots, X_n$ of random variables a martingale? State and prove the Hoeffding-Azuma inequality for a martingale $X_0, \ldots, X_n$ for which $|X_i - X_{i-1}| \le c_i$ always holds.

Let $0 < p < 1$ be constant. Show that for any $\varepsilon > 0$

$$(1-\varepsilon)\frac{n}{2\log_{1/q} n} \le \chi(G(n,p)) \le (1+\varepsilon)\frac{n}{2\log_{1/q} n}$$

holds with probability $1 - o(1)$ as $n \to \infty$, where $\chi(G)$ is the chromatic number of G, and $q = 1 - p$. (You may assume any correct bounds on $\mathbb{E}(X_k)$ and $\mathbb{E}(Y_k)$, where X_k is the number of k-cliques in $G(n,p)$ and Y_k is the number of ordered pairs of k-cliques sharing at least one edge.)

5 Let Λ be an l by l matrix with non-negative entries λ_{ij}. The branching process $B_i(\Lambda)$, $1 \le i \le l$, is defined as follows: start with a single particle of type i in generation 0. Each particle of type i has for each j a Poisson number of children of type j, with mean λ_{ij}. The numbers of children of different types of a given particle are independent, as are the children of different particles. Let p_i be the probability that $|B_i(\Lambda)| = \infty$.

(a) Show that the vector $\mathbf{p} = (p_1, \ldots, p_l)$ is the pointwise maximum non-negative solution to

$$p_i = 1 - e^{-\sum_{j=1}^{l} \lambda_{ij} p_j},$$

i.e., show that $\mathbf{p}$ is a solution, and that if $\mathbf{p}'$ is another non-negative solution, then $p_i \ge p'_i$ for every i.

(b) What is the expected size of the t^{th} generation of $B_i(\Lambda)$?

(c) State and prove a necessary and sufficient condition on Λ for $\sum_i p_i$ to be positive.

(You may wish to use the following fact: if M is a matrix with non-negative entries, then the largest eigenvalue λ of M is equal to the maximum $\alpha \ge 0$ such that there is a non-zero vector $\mathbf{v}$ with non-negative entries v_i for which $(M\mathbf{v})_i \ge \alpha v_i$ for every i.)

6 Let $G_n = G_n^{(1)}$ be the scale-free LCD graph on n vertices with $2n$ edges. Give the definition of G_t in terms of G_{t-1}.

(a) Give the standard alternative description of G_n in terms of random variables $R_1, \ldots, R_n, L_1, \ldots, L_n \in [0,1]$, and show that it is equivalent.

(b) Show that for any fixed $x > 0$, $\mathbb{P}(R_1 \geq x/\sqrt{n}) \to e^{-x^2}$ as $n \to \infty$.

(c) Let $d_1(n)$ be the degree of vertex 1 in G_n. Show that for any fixed $y > 0$,

$$\mathbb{P}(d_1(n) \geq y\sqrt{n}) \to e^{-y^2/8}$$

as $n \to \infty$.

(Hint: first show that for some $\varepsilon(n) \to 0$, for example $\varepsilon(n) = 1/\log n$, the event that $(1-\varepsilon(n))\sqrt{\frac{i}{2n}} \leq R_i \leq (1+\varepsilon(n))\sqrt{\frac{i}{2n}}$ holds for all $i \geq n^{1/10}$ has probability $1-o(1)$.)

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