UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 1.30 to 3.30

PAPER 11

GRAPHS AND HYPERGRAPHS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $r \in \mathbb{N}$ and $\epsilon > 0$. Prove that there exist $d(r,\epsilon)$ and $n_0(r,\epsilon)$ such that if G is a graph with $|G| = n \geq n_0$ and $\delta(G) \geq (1-1/r+\epsilon)n$ then $K_{r+1}(t) \subset G$, where $t = \lfloor d \log n \rfloor$.

State the Erdős-Stone theorem and explain how it follows from the result just proved.

Let F be the graph of order 6 formed by removing 3 independent edges from K_6. Let $\mathrm{ex}(n;H) = \max\{e(G) : |G| = n, F \not\subset G\}$. What is $\lim_{n\to\infty} \mathrm{ex}(n;H)/\binom{n}{2}$?

Show that $\mathrm{ex}(n;H) > t_2(n)$ for $n \geq 3$, where $t_2(n)$ is the size of the bipartite Turán graph $T_2(n)$.

2 Prove Szemerédi's Regularity Lemma. [You may assume the definition of ϵ-uniformity, that if $U' \subset U$ and $W' \subset W$ satisfy $|U'| \geq (1-\delta)|U|$ and $|W'| \geq (1-\delta)|W|$ then $|d(U',W') - d(U,W)| \leq 2\delta$, and also any quantitative form of the Cauchy-Schwarz inequality that you need.]

Let G be a graph of order n with $n^2/6$ edges. Show that there exists $c > 0$ such that $V(G)$ contains a $1/8$-uniform pair of subsets (U,W) with $d(U,W) > 1/4$ and $|U| = |W| \geq cn$.

3 *Either* prove that every oriented tree of order n is contained in every tournament of order $3n-3$, but that not every oriented tree of order n is contained in every tournament of order $2n-3$,

Or prove that every strong digraph D is spanned by $\alpha(D)$ circuits, but never by $\alpha(D)-1$ circuits. [You may assume Dilworth's theorem, and that every strong digraph has a coherent ordering.]

4 Let F be the Fano plane. Describe, making clear the main ideas, a proof that $\mathrm{ex}(n;F) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$.

END OF PAPER