UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 1.30 to 3.30

PAPER 10

EXTREMAL COMBINATORICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the vertex-isoperimetric inequality in the discrete cube (Harper's Theorem).

Let n be odd, and let $A \subset Q_n$ with $|A| = 2^{n-1}$. How small can the vertex-boundary of A be? How large can it be? What are the corresponding answers when n is even?

For any n, and any $0 \leqslant k \leqslant 2^n$, let $C(k)$ denote the initial segment of the simplicial order on Q_n of size k. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) If $k \leqslant l$ then the neighbourhood of $C(k)$ is at most as large as that of $C(l)$.

(ii) If $k \leqslant l \leqslant 2^{n-1}$ then the vertex-boundary of $C(k)$ is at most as large as that of $C(l)$.

(iii) If $k + l = 2^n$ and $k \leqslant l$ then the vertex-boundary of $C(k)$ is at least as large as that of $C(l)$.

2 What is the *standard* simplicial decomposition of S^n? What is a *regular* simplicial decomposition of S^n?

Let F^n be the standard simplicial decomposition of S^n, and let F be a regular simplicial decomposition of S^k. Show that every antipodal simplicial map $f: F \to F^n$ has a positive simplex.

State the Borsuk-Ulam Theorem, and deduce it from what you have just proved. [*If you prove an equivalent form of the Borsuk-Ulam Theorem, you must explain why the version you have proved does imply the Borsuk-Ulam Theorem. You may assume the existence and properties of barycentric subdivisions.*]

State and prove Kneser's Conjecture.

3 State and prove the Frankl-Wilson Theorem (on modular intersections).

Let p be prime. Show that if $A \subset [4p]^{(2p)}$ satisfies $|x \cap y| \neq p$ for all $x, y \in A$ then $|A| \leqslant 2\binom{4p}{p-1}$. Indicate clearly where in your proof the factor of '2' in this upper bound comes from.

Show that there does exist such a family A of size $2\binom{3p-1}{p-1}$.

4 State and prove the Uniform Covers Theorem.

State and prove the Bollobás-Thomason Box Theorem.

State the Loomis-Whitney Theorem, and deduce it from the Bollobás-Thomason Box Theorem.

For which positive reals a, b, c does there exist a body S in $\mathbb{R}^3$ of volume 1 such that $|S_{12}| = a,\ |S_{23}| = b,\ |S_{31}| = c$?

END OF PAPER