UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 1.30 to 4.30

PAPER 9

CONFORMAL MAPPINGS

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathbb{C}_\infty$ be the extended complex plane, and let φ be the stereographic projection of $\mathbb{C}_\infty$ onto the Riemann sphere Σ (the unit sphere in $\mathbb{R}^3$). Let $g(z) = (az+b)/(cz+d)$, where $ad - bc = 1$. State and prove a necessary and sufficient condition (in terms of the coefficients a, b, c and d) for $\varphi g \varphi^{-1}$ to be a rotation of Σ.

Prove that C is a circle (including a line with ∞ attached) in $\mathbb{C}_\infty$ if and only if $\varphi(C)$ is a circle on Σ.

2 Let D be a plane domain, and let $\mathcal{F}$ be a family of functions, each analytic in D. Define what is meant be the statement '$\mathcal{F}$ *is a normal family in* D' (or '*is normal in* D').

Show that $\mathcal{F}$ is normal in D if and only if each z_0 in D has a neighbourhood $N(z_0)$ such that $\mathcal{F}$ is normal in $N(z_0)$.

Suppose that $\mathcal{F}$ is normal in D, and consider the family $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ of derivatives of functions in $\mathcal{F}$. By considering $f_n(z) = \frac{1}{2}nz^2 + n$, or otherwise, show that $\mathcal{F}'$ need not be normal in D even when $\mathcal{F}$ is normal there. Now suppose that for some z_0 in D, and some positive number M, $|f(z_0)| \leqslant M$ for every f in $\mathcal{F}$. Show that the family $\mathcal{F}'$ is normal in D.

3 Discuss alternative definitions of what it means to say that a domain in the complex plane is simply connected. Select two of your possible definitions and show that they are equivalent to each other.

State the Riemann Mapping Theorem, and outline the main steps of a proof.

4 (i) Suppose that $u : D \to \mathbb{R}$ is harmonic in a plane simply connected domain D. By considering

$$g(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},$$

or otherwise, show that there is some function f that is single-valued and analytic in D with u the real part of f.

(ii) Let $A = \{z : 1 < |z| < 2\}$. Give an example of a function u that is harmonic in A, but not the real part of any function f that is single-valued and analytic in A. Verify that your example has this property.

Show that the function

$$u(x,y) = \log\frac{(x+1)^2 + y^2}{(x-1)^2 + y^2}$$

is the real part of a function that is single valued and analytic in A.

5 Let $\Delta = \{x + iy : x^2 + y^2 < 4,\ y > 0\}$, and let D be the domain obtained by removing the segment $\{iy : 0 < y \leqslant 1\}$ from Δ. Let f be a (bijective) conformal mapping of D onto the unit disc $\mathbb{D}$. Prove (giving all details) that there are distinct points α and β on $\partial\mathbb{D}$ such that if z_n is a sequence in D that is converging to $i/2$, then the sequence $f(z_n)$ accumulates at and only at, either one, or both, of the points α and β.

6 Let S be the class of maps f that are analytic and univalent in the open unit disc $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Prove that if $f \in S$ then $f(\mathbb{D})$ contains the disc $\{z : |z| < 1/4\}$.
[You may assume Green's formula.]

Show that the conclusion may fail if (i) the number $1/4$ is replaced by any larger number, or (ii) the word 'univalent' is omitted.
[You may find it helpful to consider functions of the form $ae^{bz} + c$.]