UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 1.30 to 3.30

PAPER 8

INTRODUCTION TO INTEGRABLE SYSTEMS

Questions 1, 2, 3 are divided in 3 parts. Question 4 is divided in two parts.
Seven *completed parts give full marks.*

Each part carries equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let M be a differentiable manifold of dimension n. Give the definition of a Poisson structure on M and of a Hamiltonian system.

(ii) State the Arnol'd–Liouville theorem.

(iii) Informally discuss infinite dimensional integrability.

2 Let $B = \mathcal{C}^\infty(\mathbb{R})$ and let $\partial = \frac{\partial}{\partial x}$ denote the derivative operator acting on B. For every $f \in B$, denote by $m_f : B \to B$ the operator of point–wise multiplication by f:

$$m_f : g \mapsto fg$$

(i) Compute the composition rule of $\partial^n \cdot m_f$ for every positive integer $n \in \mathbb{Z}_+$. Generalize it for any integer $n \in \mathbb{Z}$.

(ii) Denote the pseudo–differential operator of order α, $\alpha \in \mathbb{Z}$, by the formal expression

$$P := \sum_{j=0}^{\infty} g_j \partial^{\alpha - j}, \qquad g_j \in B, \quad g_0 \neq 0.$$

For every integer n, define the action of the pseudo–differential operator ∂^n on the function $\exp(kx)$, for some $k \in \mathbb{C}$, by the formula

$$\partial^n (e^{kx}) := k^n e^{kx}.$$

Using this definition and the composition rule derived in (i), compute the action of P on any power series in the parameter k of the form

$$w = k^\beta e^{kx} \sum_{i=0}^{\infty} \frac{w_i(x)}{k^i}, \qquad w_i \in B, \quad \beta \in \mathbb{C}.$$

(iii) Let P_1, P_2 be two pseudo–differential operators of orders α_1, α_2 respectively. What is the order of $P_1 \cdot P_2$? What is the order of $[P_1, P_2]$?

3 Let $B = \mathcal{C}^\infty(\mathbb{R})$ and let $\partial = \frac{\partial}{\partial x}$ denote the derivative operator acting on B. Denote the pseudo–differential operator of order α, $\alpha \in \mathbb{Z}$, by the formal expression

$$P := \sum_{j=0}^{\infty} g_j \partial^{\alpha-j}, \qquad g_j \in B, \quad g_0 \neq 0.$$

(i) Define the Adler's trace of a pseudo–differential operator. Let P_1, P_2 be two pseudo–differential operators of orders α_1, α_2 respectively. Show that

$$\mathrm{Tr}\left([P_1, P_2]\right) = 0.$$

(ii) Let

$$M_m = \left\{ L = \partial^m + \sum_{l=0}^{m-1} u_l \partial^l,\ u_l \in B \right\}$$

denote the space of differential operators of order $m \in \mathbb{Z}_+$ and leading coefficient 1. Given any pseudo–differential operator P, define the linear operator $l_P : M \to \mathbb{R}$ by

$$l_p : L \mapsto \mathrm{tr}(P \cdot L)$$

Show that

$$l_P(L) = c + \sum_{l=0}^{m-1} \int a_l(x) u_l(x) \mathrm{d}x,$$

for some $c \in \mathbb{R}$ and some $a_1, \ldots, a_{m-1} \in B$.

(iii) Let P_1, P_2 be two pseudo–differential operators of orders α_1, α_2 respectively. Let

$$M_2 = \left\{ L = \partial^2 + u,\ u \in B \right\}.$$

Define

$$\{l_{P_1}, l_{P_2}\}(L) := \mathrm{Tr}(L[P_1, P_2]), \qquad L \in M_2.$$

Compute $\{l_{P_1}, l_{P_2}\}(L)$ in the case $P_1 = f(x)\partial^{-1}$ and $P_2 = g(x)\partial^{-1}$. Compute

$$\{u(x), u(y)\}.$$

4 Let M be a differentiable manifold of dimension n and $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ denote two Poisson structures on M such that

$$\{\cdot,\cdot\}_\lambda := \{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$$

is again a Poisson structure on M, $\forall\lambda \in \mathbb{C}$.

(i) Let

$$H = \sum H_n \lambda^n, \qquad H_n \in M,$$

denote a formal series in $\lambda \in \mathbb{C}$. Suppose that H is a Casimir for the Poisson structure $\{\cdot,\cdot\}_\lambda$. Show that

$$\{H_0, f\}_1 = 0, \qquad \forall f \in \mathcal{C}^\infty(M),$$

$$\{H_n, f\}_1 = -\{H_{n-1}, f\}_2, \qquad \forall f \in \mathcal{C}^\infty(M), \quad \forall n \geqslant 1,$$

and

$$\{H_n, H_m\}_1 = \{H_n, H_m\}_2 = 0, \qquad \forall n, m \geqslant 0.$$

(ii) Given a Casimir H_0 of the first Poisson structure $\{\cdot,\cdot\}_1$, show how to recursively define an infinite family of functions H_n in involution with respect to $\{\cdot,\cdot\}_\lambda$, $\lambda \in \mathbb{C}$.