UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 1.30 to 3.30

PAPER 79

SEISMIC WAVES

Any number of questions may be attempted.

Full marks can be obtained for one complete answer or its equivalent.

There are **three** *questions in total.*

The questions carry equal weight.

Candidates may use their lecture notes, any material handed out during the course and examples classes, and any hand-written or typed notes, taken from sources outside the lectures, which they have prepared themselves.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe and explain the transmission and reflection properties of one-dimensional seismic waves for a composite medium composed of many parallel uniform layers. A substantial answer to this question will draw on your own investigation of these properties and your experience using the spread-sheet program provided during the course for one-dimensional waves in such media imbedded between uniform half-spaces.

2 The interface between a solid, with density ρ and P and S wave speeds α and β, and overlying fluid, with density ρ_f and P wave speed α_f, is the cylindrically symmetric hyperbolic surface

$$z = z_0\left(\frac{\alpha}{\alpha+\alpha_f}\right) + \left(\frac{\alpha_f}{\alpha^2-\alpha_f^2}\right)\sqrt{z_0^2\left(\alpha-\alpha_f\right)^2 + R^2\left(\alpha^2-\alpha_f^2\right)},$$

where z is positive in the upwards direction, $z_0 > 0$, $\alpha > \alpha_f$, and $R^2 = x^2 + y^2$. There is a point source of P waves at the origin $(x, y, z) = (0, 0, 0)$ in the solid. The radiated displacement $\mathbf{u} = \boldsymbol{\nabla}\phi$ is associated with a scalar potential of the form

$$\phi = \tfrac{1}{r} f\left(t - \tfrac{r}{\alpha}\right),$$

where $r^2 = x^2 + y^2 + z^2$, $f(t) = 0$ for $t < 0$, and $f(t)$ is continuous and has continuous first and second derivatives at $t = 0$.

(i) Establish that the angles θ_z between the normal to the interface and the vertical axis are given by

$$\tan\theta_z = \frac{R\alpha_f}{\sqrt{z_0^2(\alpha-\alpha_f)^2 + R^2\left(\alpha^2-\alpha_f^2\right)}},$$

and hence that the angles of incidence θ_P at the interface of the P waves are given by

$$\tan\theta_P = \frac{R\alpha}{z_0(\alpha-\alpha_f)}.$$

(ii) From this deduce that the corresponding angle of refraction θ_f of the transmitted P waves in the fluid is $\theta_f = \theta_z$, and hence that the wavefronts of the transmitted P waves have the shape of plane waves propagating in the vertical direction with the travel time from the source

$$\tau = \frac{z_0}{\alpha} + \frac{(z-z_0)}{\alpha_f}$$

being independent of R.

(iii) For a plane P wave

$$\phi = f\left(t - \tfrac{x}{\alpha}\sin\theta_P - \tfrac{z}{\alpha}\cos\theta_P\right)$$

incident at a flat interface $z = 0$ between a solid and a fluid with the above properties, use the continuity of u_z, σ_{xz} and σ_{zz} to deduce that the coefficient T_P for the transmitted P wave

$$\phi = T_P f\left(t - \tfrac{x}{\alpha_f}\sin\theta_f - \tfrac{z}{\alpha_f}\cos\theta_f\right)$$

in the fluid is given by

$$T_P = \frac{2\rho\alpha_f\cos 2\theta_S\cos\theta_P}{\rho_f\alpha_f\cos\theta_P + \rho\alpha\left[\cos^2 2\theta_S + \frac{\beta^2}{\alpha^2}\sin 2\theta_P\sin 2\theta_S\right]\cos\theta_f},$$

where θ_S is the angle of the reflected S wave in the solid.

(iv) Explain the steps involved in using the above results to calculate the amplitude of the leading order wavefield discontinuity in the fluid as a function of R. Obtain expressions for this amplitude at $R = 0$ and accurate to $O\left(R^{-2}\right)$ for $R \gg z_0$.

3 (i) For harmonic waves of the form

$$e^{i(kx-\omega t)\pm ik_\beta z}$$

where k is real-valued, ω is complex-valued with $\mathrm{Im}\,(\omega) > 0$ (to move the contour of integration above the real axis for causal wavefields), and

$$k_\beta = \sqrt{\tfrac{\omega^2}{\beta^2} - k^2}$$

with real-valued wave speed β, show that the condition $\mathrm{Im}\,(k_\beta) > 0$ is equivalent to $\mathrm{Re}\,(p_\beta) > 0$, where

$$p_\beta = \tfrac{k_\beta}{\omega} = \sqrt{\tfrac{1}{\beta^2} - p^2} \quad \text{and} \quad p = \tfrac{k}{\omega}.$$

(ii) For SH waves with particle velocity and vertical traction of the forms

$$v_y = V\,(z)\,e^{i\omega(px-t)}, \quad \tau_{yz} = T\,(z)\,e^{i\omega(px-t)}$$

in a medium with density $\rho\,(z)$, S wave speed $\beta\,(z)$ and shear modulus $\mu = \rho\beta^2$ depending only on the depth z, establish that the Riccati equation satisfied by the scalar impedance $Z\,(z)$ such that $T\,(z) = -Z\,(z)\,V\,(z)$ is

$$\tfrac{dZ(z)}{dz} = -i\omega\left(\tfrac{Z(z)^2}{\mu} - \mu p_\beta^2\right),$$

and that the corresponding one-way equation satisfied by $V\,(z)$ is

$$\tfrac{dV(z)}{dz} = i\omega\left(\tfrac{Z(z)}{\mu}\right)V\,(z)\,.$$

(iii) The sign of the time average $S_z\,(z) = -\frac{1}{4}\,(V^*T + VT^*)$, where the superscript $*$ denotes the complex conjugate, determines whether energy is propagating down or up. Prove that in general $S_z\,(z)$ decreases strictly monotonically with depth (i.e. $dS_z/dz < 0$) when $\mathrm{Im}\,(\omega) > 0$ and $k = p\omega$ is real-valued. Hence, establish that if $\mathrm{Re}\,(Z\,(z_1)) > 0$ then $\mathrm{Re}\,(Z\,(z)) > 0$ for all $z < z_1$, and that if $\mathrm{Re}\,(Z\,(z_0)) < 0$ then $\mathrm{Re}\,(Z\,(z)) < 0$ for all $z > z_0$. For $\mathrm{Re}\,(p_\beta) > 0$ the two constant solutions $Z\,(z) = \pm\mu p_\beta$ to the Riccati equation in a uniform region give energy flow down and up respectively. Show that the corresponding solutions to the one-way equation are

$$V\,(z) \propto e^{\pm i\omega p_\beta z} = e^{\pm ik_\beta z},$$

and involve exponential decay in the vertical direction of energy flow.

(iv) Solve the Riccati equation in a uniform region $z \in (z_0, z_1)$ for a general initial condition, either $Z\,(z_0) = Z_0$ or $Z\,(z_1) = Z_1$, and show that the corresponding solution to the one-way equation is of the form

$$V\,(z) = V_+ e^{i\omega p_\beta z} + V_- e^{-i\omega p_\beta z}.$$

(v) For a sequence of uniform layers between $z = z_a$ and $z = z_b$ ($z_a < z_b$) with uniform half-spaces on either side, describe how you use the continuity of $Z\,(z)$ to obtain the value of $Z(z_a)$ given the value of $Z(z_b)$. Explain why $Z(z_b) = \mu p_\beta$ is the appropriate value of $Z(z)$ in the half-space below z_b when there is a downgoing incident wave

$$V(z) = V_0 e^{i\omega p_\beta (z - z_a)}$$

in the half-space above z_a. The upgoing reflected wave in the half-space above z_a will be of the form

$$V(z) = R V_0 e^{-i\omega p_\beta (z - z_a)}.$$

Show that the reflection coefficient R can be determined from the value of $Z(z_a)$ and the value of μp_β for the half-space above z_a, implying that $Z(z_a)$ contains all the information needed about reflections from the underlying layer boundaries.