MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 9.00 to 12.00

PAPER 76

ASYMPTOTIC METHODS IN FLUID DYNAMICS

Candidates may attempt **ALL** *questions.*

There are **five** *questions in total.*

The questions are of equal weight.

A distinction mark may be obtained by good answers to approximately **THREE** *questions.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) The integral $J(\lambda;\varepsilon)$ is defined for real $\lambda > 0$ by

$$J(\lambda;\varepsilon) = \int_0^1 \frac{1}{(\lambda(x-1)+1)^2 + \varepsilon^2\cos^2(\pi x/2)}\,\mathrm{d}x\,.$$

If $0 < \varepsilon \ll 1$ calculate the leading-order asymptotic approximation of $J(\lambda;\varepsilon)$ for $0 < \lambda < 1$, λ close to 1 (where how close to one should be specified), and $\lambda > 1$. Briefly discuss whether the approximation for $\lambda > 1$ is uniformly valid for $\lambda \gg 1$.

(b) In the following, the Fourier Transform is defined to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \exp(-\mathrm{i}kx) f(x)\mathrm{d}x\;.$$

(i) Show, being careful to explain each step in your reasoning, that the Fourier Transform of the Heaviside step function is

$$\pi\delta(k) + \frac{\mathrm{i}}{k}\;.$$

Hence, find the Fourier Transform of $1/x$.

(ii) Show, by using your answer to (i), or otherwise, that the Fourier Transform of $\log|x|$ is

$$-\frac{\pi}{|k|} + C\delta(k)\;,$$

where C is a constant which need not be determined.

(iii) Using your answers to (i) & (ii), and quoting without proof any standard results concerning Fourier Transforms that you require, solve for $f(x)$ the integral equation

$$⨍_{-\infty}^{\infty} \frac{f(t)}{x-t}\mathrm{d}t = \mathrm{H}(x-1) - \mathrm{H}(x+1)\;.$$

(iv) Determine the first **two** terms in the large-wavenumber expansion of the Fourier Transform of the modified Bessel function $\mathrm{K}_0(|x|)$.

[*Hint: Note that* $\mathrm{K}_0(z)$ *is the solution of the equation*

$$z^2\frac{\mathrm{d}^2\mathrm{K}_0}{\mathrm{d}z^2} + z\frac{\mathrm{d}\mathrm{K}_0}{\mathrm{d}z} - z^2 K_0 = 0$$

such that $\mathrm{K}_0(z) = -f(z)\log(z) + g(z)$*, where* f *and* g *are analytic functions, with* $f(0) = 1$*, and such that* $\mathrm{K}_0(|z|) \to 0$ *as* $|z| \to \infty$.]

2

(i) The integral $I(\lambda, n)$ is defined for real integer n and $Re(\lambda) > 0$ by

$$I(\lambda, n) = \int_0^\infty \frac{t^{n-1}\exp(\lambda(1-t))}{1-t}\,\mathrm{d}t\,,$$

where the contour of integration is assumed to pass just above the pole at $t = 1$. Suppose that $n \gg 1$ and that

$$\lambda \sim n + \mathrm{i}\mu n^{\frac{1}{2}} + \nu + \dots\,,$$

where $\mu = O(1)$ and $\nu = O(1)$. By considering $\frac{\partial I}{\partial \mu}$ or otherwise, find the leading-order asymptotic expansion for $I(\lambda, n)$ as $n \to \infty$.

(ii) Show that, formally,

$$I(\lambda, n) \sim \exp(\lambda) \sum_{r=n}^{\infty} \frac{\Gamma(r)}{\lambda^r}\,.$$

(iii) The asymptotic series for the Airy function, $\mathrm{Ai}(z)$, when $|z| \gg 1$ and $|\arg(z)| < \pi$ is given by

$$\mathrm{Ai}(z) = \frac{1}{2z^{\frac{1}{4}}\pi^{\frac{1}{2}}} \exp\left(\tfrac{1}{2}\sigma\right) \sum_{r=1}^{\infty} Y_r\,,$$

where

$$Y_r = \frac{\Gamma(r+\frac{1}{6})\Gamma(r+\frac{5}{6})}{2\pi\sigma^r\Gamma(r+1)} \quad \text{and} \quad \sigma = -\tfrac{4}{3}z^{\frac{3}{2}}\,.$$

Show that

$$Y_r \to \frac{\Gamma(r)}{2\pi\sigma^r} \quad \text{as} \quad r \to \infty\,.$$

Comment. You may quote Stirling's asymptotic approximation for the Gamma function

$$(r-1)! = \Gamma(r) \sim \left(\frac{2\pi}{r}\right)^{\frac{1}{2}} r^r\,\mathrm{e}^{-r} \quad \text{as} \quad r \to \infty\,.$$

(iv) Hence show that if the asymptotic series for the Airy function is truncated at the smallest term, say $r = n$, then for

$$\arg(\sigma) = \frac{\phi}{|\sigma|^{\frac{1}{2}}} \quad \text{where} \quad \phi = O(1)\,,$$

the remainder R_n is given by

$$R_n \sim \frac{\mathrm{i}\exp\left(-\frac{1}{2}\sigma\right)}{4z^{\frac{1}{4}}\pi^{\frac{1}{2}}} \left(1 + \mathrm{erf}\left(\frac{\phi}{2^{\frac{1}{2}}}\right)\right)$$

Briefly interpret this result.

3 The function $y(x)$ satisfies the differential equation

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + xy + y^2 = 0\,.$$

If $y(a) = \alpha$ and $y(1) = \beta$, with $0 < a < 1$, identify for what values of α and β solutions can be found in the limit $\varepsilon \to 0$ on the assumption that there are no rapid oscillations or 'internal' boundary layers away from the end points. Sketch your solution[s] indicating whether there is a unique solution. Briefly discuss whether additional solutions with internal boundary layers can be easily ruled out.

Next suppose that $a = \alpha = 0$, i.e. $y(0) = 0$. Derive the governing equation in the inner region near $x = 0$ and state the boundary conditions a solution should satisfy. Without solving the equation exactly, discuss whether a solution satisfying the boundary conditions is likely to exist, e.g. on the basis of linearising the equation in the 'intermediate matching region' and discussing the linearised equation's solutions.

Comment. You may quote the result that

$$\int \frac{\mathrm{d}y}{\sqrt{a^3 - 3ay^2 - 2y^3}} = -\frac{2}{\sqrt{3a}} \operatorname{arctanh}\left(\sqrt{\frac{a-2y}{3a}}\right)\,.$$

4

(a) (i) Consider the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \epsilon c(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + x = k \ ,$$

subject to $x = \mathrm{d}x/\mathrm{d}t = 0$ when $t = 0$, where c, k and ϵ are real positive constants with $\epsilon \ll 1$. Use the method of multiple scales to find the leading-order approximation to $x(t;\epsilon)$ which is uniformly valid for $t \leq O(1/\epsilon)$.

(ii) Now consider the coupled system

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \epsilon y(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + x = k \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \epsilon(1+x-y)\end{aligned}$$

such that $x = \mathrm{d}x/\mathrm{d}t = y = 0$ when $t = 0$. Find the uniformly-valid leading-order approximations for $x(t;\epsilon)$ and $y(t;\epsilon)$ when $t \leq O(1/\epsilon)$.

(b) The non-dimensional equations describing the inviscid one-dimensional motion of a gas are

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \frac{\partial u}{\partial x} + M\frac{\partial(\rho u)}{\partial x} &= 0 \\ (1+M\rho)\left(\frac{\partial u}{\partial t} + Mu\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} &= 0 \\ (1+M\gamma p) &= (1+M\rho)^\gamma \ ,\end{aligned} \tag{1}$$

where p, ρ and u are the unsteady pressure and density perturbation and the velocity respectively, γ is a positive constant and $M \ll 1$ is the constant Mach number. By writing

$$p(x,t;M) = p_0(\theta,X) + Mp_1(\theta,X) + ... \ ,$$

with similar expansions for p and u, where $X = Mx$ and $\theta = t - x$, show that p_0 satisfies

$$\frac{\partial p_0}{\partial X} - \left(\frac{\gamma+1}{2}\right)p_0\frac{\partial p_0}{\partial\theta} = 0 \ , \tag{2}$$

uniformly in $|\theta| \leq O(M^{-1})$.

How does equation (2) change if the viscous term $\beta\partial^2u/\partial x^2$, with $\beta = O(M)$, is introduced onto the right-hand side of equation (1)?

5

(a) (i) Describe the use of the Briggs-Bers technique to determine the long-time behaviour of solutions of the one-dimensional linearised Ginzburg-Landau equation with constant real coefficients,

$$\mathcal{L}\eta \equiv \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} - \mu\eta - \gamma\frac{\partial^2 \eta}{\partial x^2} = 0 \ .$$

Find the conditions involving U, μ and γ for the occurrence of convective instability and of absolute instability.

(ii) By taking Fourier transforms in both x and t, find the solution of

$$\mathcal{L}\eta = \delta(x)\delta(t) \ .$$

In the limit $t \to \infty$ relate the behaviour of your solution to the conditions found in (i) above.

(iii) In two dimensions the linearised Ginzburg-Landau equation with constant coefficients is

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} + V\frac{\partial \eta}{\partial y} - \mu\eta - \gamma_{11}\frac{\partial^2 \eta}{\partial x^2} - \gamma_{22}\frac{\partial^2 \eta}{\partial y^2} - 2\gamma_{12}\frac{\partial^2 \eta}{\partial x \partial y} = 0 \ ,$$

where U, V, μ and γ_{ij} are real constants. By considering solutions of the form $\eta = \exp(-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}ly)$ and computing the group velocity, $\left(\frac{\partial \omega}{\partial k} \ , \ \frac{\partial \omega}{\partial l}\right)$, find a necessary condition for the occurrence of absolute instability in the form

$$\mu > f(U, V) \ ,$$

where $f(U, V)$ is a quadratic function of U, V to be determined.

Given that the vector (U, V) represents a mean-flow velocity whose magnitude is fixed but whose orientation can be changed, find the minimum value of μ required to guarantee the absence of absolute instability for **any** mean-flow direction.

(b) Consider the linearised Ginzburg-Landau equation in one dimension on the semi-infinite domain $x > 0$, with boundary conditions $A = 0$ at $x = 0$ and $A \to 0$ as $x \to \infty$. The coefficients U and γ are real constants, but $\mu = \mu_0 - \epsilon\lambda x$ with $\epsilon \ll 1$, and μ_0 and λ positive real constants.

By considering a solution of the form

$$f(\epsilon^\sigma x) \exp\left(\frac{Ux}{2\gamma} - \mathrm{i}\omega t\right) \ ,$$

where the index $\sigma > 0$ and the function f are to be determined, show that the global mode frequencies are

$$\mathrm{i}\left(\mu_0 - \frac{U^2}{4\gamma} + (\epsilon^2\gamma\lambda^2)^{\frac{1}{3}} z_n\right) \qquad n = 1, 2, 3, \ldots.. \ ,$$

where z_n is the nth zero of the Airy function $\mathrm{Ai}(x)$ on the negative real axis. Hence, deduce the condition for global instability, and compare this to the condition for local absolute instability.

[*Hint:* $\mathrm{Ai}(x)$ *is the solution of* $y'' = xy$ *which is bounded as* $x \to \infty$. Ai *has a countable number of real zeros, all negative.*]