UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 9 to 12

PAPER 75

ENVIRONMENTAL FLUID DYNAMICS

Attempt **BOTH** *questions in Section A and* **THREE** *in Section B.*

There are **six** *questions in total.*

Questions in Section A carry half the weight of questions in Section B.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 A long wave tank contains water initially at rest with a uniform depth H. A piston-like wave maker located at $x = 0$ is turned on at $t = 0$ inducing a depth averaged velocity $U \sin \omega t$ at this point.

(a) Derive the dispersion relationship for small amplitude waves and determine the phase and group velocities. Describe the deep water and shallow water limits.

(b) For finite amplitude waves in the shallow water limit, determine the depth of the water at the wave maker. Use an $x - t$ diagram and sketches to outline the evolution of the waves as they propagate along the tank. Show that the leading edge of the first wave produced forms a shock (bore) at a distance $\frac{4}{9}\frac{gH}{U\omega}$ from the wave maker.

2 (a) Describe the "4/3's" heat transfer law for high Rayleigh number Rayleigh-Benard convection, stating all assumptions.

(b) A horizontal layer of fluid lies between two horizontal boundaries separated by a distance h. For times $t < 0$ the fluid layer and the upper and lower boundaries are at temperature $T = T_o$. At time $t = 0$, the temperature of the lower boundary is impulsively raised to $T = T_o + \Delta T$ with $\Delta T > 0$ and then maintained at this new temperature. Assuming that the "4/3's" law can be used at all times to model the heat transfer across the upper and lower boundaries, derive (BUT DO NOT SOLVE) an equation to model the evolution of the mean temperature of the fluid layer, stating all assumptions.

(c) Assuming that the "4/3's" law can be replaced by a "3/3's" law, determine the mean temperature in the fluid layer as a function of time. What is the mean temperature in the fluid layer in the limit as $t \to \infty$?

Section B

3 A fluid layer of depth h and velocity u contains a dilute suspension of particles of density ρ_p, volume concentration ϕ and settling velocity V_s in fluid of density ρ_0. The particle-laden layer lies below a deep ambient layer of density ρ_0 in a channel of unit width.

(a) Describe briefly how turbulence of intensity u' can keep the particles in suspension. Why are particles still lost to a lower boundary under such conditions?

(b) Derive the shallow water equations for the case of a layer of constant volume that remains vertically well mixed. Determine the characteristics of the flow, and the ordinary differential equations for the properties along these characteristics. You may neglect any drag terms.

(c) At time $t = 0$ a layer with particle concentration ϕ_o is confined by a barrier to $x < 0$ where it has uniform depth H. The barrier is removed at $t = 0$, allowing a gravity current to form. The motion of the front of this current is described by a Froude number F. Outline the origin of this front condition and the assumptions that go into its derivation (you need not derive the condition). Does the shallow water approximation apply at the front?

(d) In the limit of $V_s = 0$, determine the speed of the front and show that for finite F a region of uniform depth exists behind the front. Describe the evolution of the current, giving profiles of the depth h and velocity u.

(e) Find an explicit expression for the location of the leading edge of the rarefaction wave when $V_s \neq 0$. Using the approximation that the depth of the current near the front remains at the $V_s = 0$ value, find also the location of the front when $V_s \neq 0$.

4 A thermal is an instantaneous release of buoyancy from a point source. At time $t = 0$ a thermal of total buoyancy B_o is released from a source at height $z = 0$. The thermal can be modelled as a sphere of radius b and density ρ rising with velocity u in a quiescent ambient of uniform density ρ_o according to

$$\frac{d}{dt}\left(\frac{4}{3}\pi b^3\right) = 4\pi b^2 \alpha u\,,$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi b^3 \rho u\right) = \frac{4}{3}\pi b^3 g(\rho_o - \rho)\,,$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi b^3 g\frac{(\rho_o - \rho)}{\rho_{ref}}\right) = 0\,,$$

$$\frac{dz}{dt} = u\,,$$

where α is the turbulent entrainment coefficient, g is the acceleration of gravity and ρ_{ref} is a reference density.

(a) Give a physical interpretation of each of the above equations. Describe how the Boussinesq approximation can be used to simplify these equations and hence solve for $b(t)$, $u(t)$, $z(t)$ and $\rho(t)$ when the ambient density ρ_o is uniform.

(b) Why does a thermal released in an ambient with a constant stable stratification with buoyancy frequency N have a finite rise height? Discuss how dimensional analysis can be used to estimate the height of rise of the thermal.

(c) Suppose the ambient fluid is replaced by a two-layer stratification in a cubic enclosure with cross-sectional area A. The lower layer has depth h_1 and density ρ_1 and the upper layer has depth h_2 and density ρ_2 with $\rho_2 < \rho_1$. From time $t = 0$, thermals are released from a source at the base of the enclosure at a constant rate f and for $0 < t < t^*$ these thermals impinge upon the density interface separating the layers in such a way that fluid from the upper layer is entrained into the lower layer in a quasi-steady manner. It can be assumed that the ambient fluid in the lower layer remains well mixed at all times. At time $t = t^*$ the buoyancy in the thermal at the height of the interface is such that the thermal is able to pass through the interface and rise to the top of the domain. The rate of entrainment across the density interface can be modelled as

$$E = \frac{dh_1}{dt}/u^* = cRi^{-n}\,,$$

where $Ri = g'b^*/u^{*2}$ is the Richardson number and c and n are numerical constants. The Richardson number is defined in terms of the reduced gravity g' across the density interface and the thermal radius b^* and velocity u^* at the height of the interface. Give a physical interpretation of the Richardson number and comment on physically meaningful values of c and n. Assuming that $c = n = 1$ calculate $h_1(t)$ for $0 < t < t^*$. What is the density of the lower layer at the time t^* when the first thermal is able to pass through the density interface?

5 (a) Consider an inviscid two-dimensional stratified flow in the $x-z$ plane described by a constant buoyancy frequency N. Write down the equations of motion and show that these can be linearised about a state of rest to obtain

$$\nabla \cdot \mathbf{u}' = 0,$$

$$\frac{\partial \mathbf{u}'}{\partial t} = -\frac{1}{\rho_0}\nabla p' - \sigma\hat{\mathbf{z}},$$

$$\frac{\partial \sigma}{\partial t} = -w' N^2,$$

where $\mathbf{u}' = (u', w')$ is the velocity perturbation, $\hat{\mathbf{z}}$ is the unit vector in the z direction, p' is the pressure perturbation, ρ_0 is the reference density, $\sigma = -\frac{g}{\rho_0}\rho'$ is the buoyancy perturbation and g is gravity. Hence or otherwise derive the dispersion relation for small amplitude plane waves. Show that the phase and group velocities are perpendicular. Give a physical interpretation for the phase and group velocities.

(b) The stratified fluid flows with a uniform horizontal velocity U in the x-direction over a sinusoidal topography described by $z = \eta$, where $\eta = \eta_0 \cos \alpha x$, α is the wavenumber and η_0 the amplitude of the topography. Sketch the characteristics, phase and group velocities for the waves generated by the topography in both the frame of reference of the fluid and the frame of reference of the topography. State any assumptions made.

(c) Suppose the stratified fluid has a flat, rigid upper boundary at height H above the topography. Sketch and describe the modified wave field. Determine the amplitude and phase of the upward propagating wave field. Show that resonance occurs when $kH \tan\theta = n\pi$ for some integer n, where θ is the angle between the group velocity and vertical.

6 Consider an axisymmetric turbulent plume rising in a uniform ambient from a source located at height $z = 0$. The plume has volume flux πQ, specific momentum flux πM and specific buoyancy flux πB. Assuming top-hat plume variables, $Q = b^2 w$, $M = b^2 w^2$ and $B = b^2 w g'$, where b is the plume radius, w is the vertical velocity in the plume and g' is the reduced gravity within the plume. The governing plume equations may be written as

$$\frac{dQ}{dz} = 2\alpha M^{1/2},$$

$$\frac{dM}{dz} = \frac{QB}{M},$$

$$\frac{dB}{dz} = 0,$$

where α is the entrainment coefficient.

(a) Discuss the entrainment hypothesis for turbulent plumes. For a pure plume with $B = B_o > 0$, $Q = 0$, $M = 0$ at $z = 0$, solve the plume equations for Q, M, and B as a function of z and hence determine b, w and g'. Show that

$$\Gamma = \frac{M^{5/2}}{BQ^2}$$

is constant.

(b) Consider the case of a forced plume with $Q = Q_o > 0$ and $M = M_o > 0$ at $z = 0$. Define two lengthscales which result from introducing finite Q_o and M_o and discuss their physical significance. Show that

$$\frac{1}{Q_o}\frac{dQ}{dz} = \left(\frac{20\alpha^4 B_o}{Q_o^3}\right)^{1/5}\left(\frac{Q^2}{Q_o^2} - c\right)^{1/5},$$

where c is a numerical constant which is to be determined. What is the physical significance of the cases $c < 0$, $c = 0$ and $c > 0$? Show that for all z, M and Q satisfy

$$\frac{8\alpha M^{5/2}}{5B_o Q_o^2} + c = \frac{Q^2}{Q_o^2}.$$

(c) Show that

$$\frac{db}{dz} = \frac{6\alpha}{5} - \frac{cQ_o^2 B_o}{2M^{5/2}}$$

and comment on how the rate of spread of the plume varies (i) as a function of c and (ii) in the limit as $z \to \infty$. Under what conditions can necking of the plume occur (i.e. $\left.\frac{db}{dz}\right|_{z=0} < 0$)?