UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 1.30 to 4.30

PAPER 74

SLOW VISCOUS FLOWS

Attempt at most **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Substantially complete answers will be viewed more favourably than fragments.

A Distinction mark can be gained by substantially complete answers to two questions.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The Papkovich–Neuber representation of Stokes flow is

$$\mathbf{u} = \boldsymbol{\nabla}(\mathbf{x}\cdot\boldsymbol{\Phi} + \chi) - 2\boldsymbol{\Phi}, \;\; p = 2\mu\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} \text{ where } \nabla^2\chi = 0, \;\; \nabla^2\boldsymbol{\Phi} = \mathbf{0}.$$

Find the velocity $\mathbf{u}(\mathbf{x})$, vorticity $\boldsymbol{\omega}(\mathbf{x})$ and strain rate $\mathbf{e}(\mathbf{x})$ in the Stokes flow due to a point force $\mathbf{F}$ acting at the origin of an unbounded fluid of viscosity μ.

A force-free couple-free rigid sphere of radius a is placed in an unbounded strain flow with uniform strain rate $\mathbf{E}$. Find the perturbation to the flow arising from the presence of the sphere.

Two rigid spheres of radius a are placed far apart in unbounded fluid, which is otherwise at rest. The first sphere is acted on by a force $\mathbf{F}$ and is couple free. The second sphere is force free and couple free. Explain why the first sphere moves with velocity

$$\mathbf{U} = \mathbf{U}_0 - \frac{15a^4}{4R^6}(\mathbf{U}_0\cdot\mathbf{R})\mathbf{R} + O(a^5/R^5),$$

where $\mathbf{U}_0$ is the velocity that the first sphere would have if the second sphere were absent, and $\mathbf{R}$ is the vector distance between the centres of the spheres.

By considering $\mathbf{F}\cdot\mathbf{U}$, explain why this result is consistent with the Minimum Dissipation Theorem.

Find the change in velocity of the first sphere if the second sphere is still force free, but now prevented from rotating by a suitable couple.

[*You may quote the results that the Stokes drag on a translating sphere is* $6\pi\mu a U$ *and the Stokes flow round a sphere rotating with angular velocity* $\boldsymbol{\Omega}$ *is* $\mathbf{u} = \boldsymbol{\Omega}\wedge\mathbf{x}a^3/r^3$.]

2 Use the lubrication approximation to show that the thickness $h(x,t)$ of two-dimensional gravity-driven flow of a thin layer of viscous fluid on an inclined plane of slope $\theta \ll 1$ is described by the equation

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left[h^3\Big(\frac{\partial h}{\partial x} - \theta\Big)\right], \qquad (1)$$

where x is the downslope coordinate, there is no cross-slope variation and surface-tension is assumed to be negligible. Find the condition for the neglect of inertia if the typical thickness and downslope lengthscales are $\hat{h}$ and $\hat{x}$ respectively, distinguishing between the cases $\hat{h} \ll \theta\hat{x}$ and $\hat{h} \gg \theta\hat{x}$.

Consider the release of a fixed area A (i.e. volume per cross-slope distance) of viscous fluid at $x = 0$. After a long time the nose of the current has travelled a distance $x_N(t) \gg (A/\theta)^{1/2}$. Find an approximate similarity solution for the thickness $h(x,t)$ and determine $x_N(t)$ from mass conservation.

The similarity solution ends abruptly at $x_N(t)$ with a front of height $h_N(t)$. Show that

$$\frac{dx_N}{dt} = \frac{g\theta h_N^2}{3\nu}\,.$$

Re-examine the behaviour near the nose by making the change of variable $y = x - x_N(t)$ in equation (1). Hence show that the shape of the nose is quasi-steady and given by

$$\frac{\theta y}{h_N} = \frac{h}{h_N} + \frac{1}{2}\ln\left(\frac{h_N - h}{h_N + h}\right) \qquad (-\infty < y \le 0).$$

Sketch this solution and show that the lubrication approximation breaks down in an $O(\theta^{1/2}h_N)$ neighbourhood of $y = 0$.

Use scaling arguments to estimate the value of h_N at which surface tension would become as important as gravity in determining the length of the nose.

3 State and prove the Reciprocal Theorem for two Stokes flows with viscosity μ and no body force.

Prove that the resistance matrix, giving the force $\mathbf{F}$ and couple $\mathbf{G}$ exerted *by* a rigid body when moving with velocity $\mathbf{U}$ and angular velocity $\boldsymbol{\Omega}$ through surrounding viscous fluid, is both symmetric and positive definite.

A rigid body comprises three point masses with weights mg at $O = (0,0,0)$, λmg at $A = (2L,0,0)$ and λmg at $B = (0,2L,0)$, joined along OA and OB by two thin rods of negligible weight, length $2L$ and thickness ϵL. The hydrodynamic resistance to motion of the point masses is negligible and that of the thin rods is given by the slender-body formula

$$\mathbf{f}(\mathbf{X}) = C(\mathbf{I} - \tfrac{1}{2}\mathbf{X}'\mathbf{X}') \cdot \dot{\mathbf{X}}$$

where $C = 4\pi\mu/|\ln\epsilon|$ and $\mathbf{X}(s,t)$ is the position along the rod. Calculate the 6×6 resistance matrix for this body with respect to the axes fixed in the body. Deduce that

$$CL \begin{pmatrix} U_x \\ U_y \\ \Omega_z L \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{4} \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ G_z/L \end{pmatrix}.$$

The body is allowed to fall under gravity from an initial position in which OA is horizontal and B is vertically above O. Show that if $\lambda = 1$ the body falls vertically without rotation, but if $\lambda < 1$ the angle $\theta(t)$ that OA makes above horizontal increases monotonically from 0 to $\pi/4$ as $t \to \infty$. What happens if $\lambda > 1$?

Show further that if $\lambda < 1$ then the point O drifts sideways as it falls by a total horizontal distance $2L/3$ in the direction of the initial orientation of OA. Find the corresponding result for $\lambda > 1$.

4 Use scaling arguments to justify the lubrication approximation of the Navier–Stokes equations for a thin fluid film of typical thickness H and lengthscale $L \gg H$ flowing with typical velocity U over a rigid surface.

A thin film of fluid of viscosity μ and thickness $\epsilon\bar{h}(\bar{z},\bar{t})$, where $\epsilon \ll 1$ and $\bar{z}$ is the axial coordinate, coats the outside of a rigid cylinder of radius a. The free surface is acted on by uniform surface tension γ, and gravity is negligible. Derive the dimensionless evolution equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial z}\left[h^3\Big(\frac{\partial h}{\partial z} + \frac{\partial^3 h}{\partial z^3}\Big)\right] = 0, \tag{2}$$

giving the definition of the various dimensionless variables.

Find the (dimensionless) growth rate s of small disturbances of wavenumber k to a film of uniform thickness h_0, and sketch $s(k)$. What is the most unstable wavelength? Comment on the qualitative difference between this result and the most unstable wavelength for the Rayleigh instability of a viscous cylinder of fluid.

After the instability has developed for a long time it is observed that almost all the fluid collects into 'collars' around the cylinder, which are separated by very thin films of fluid. Somewhat surprisingly, the collars are found to slide slowly along the cylinder, leaving an even thinner film behind them than the film they are advancing over. Model this phenomenon as follows:

Consider an approximate solution to (2) of the form $h(z,t) = h(x)$, where $x = z - ct$, $h \to h_\pm$ as $x \to \pm\infty$, $0 < h_- < h_+ \ll 1$ and $0 < c \ll 1$. Show that the (quasi-steady) shape of a collar, where $h = O(1)$, is given by

$$h = A(1 + \cos x) + B\ . \tag{3}$$

(Here B is assumed to be a small constant and A grows slowly at a rate $c(h_+ - h_-)$.)

Near $x = \pm\pi$ there are small regions where the curvature changes rapidly to match the edges of the collar to the uniform films ahead and behind. Show that it is possible to rescale the variables in each of these regions in such a way that the leading-order equation becomes $H^3 H''' = H - 1$. Explain why this equation has, to within translations in X, a unique solution $H_-(X)$ with $H_- \to 1$ as $X \to -\infty$ and a one-parameter set of solutions $H_+(X;\lambda)$ with $H_+ \to 1$ as $X \to +\infty$.

If $H_- \sim a_- X^2 + b_-$ as $X \to \infty$ and $H_+ \sim a_+(\lambda)X^2 + b_+(\lambda)$ as $X \to -\infty$ show, by matching to (3), that

$$c = (Ah_+/2\alpha)^{3/2}, \quad B = h_+\beta \quad \text{and} \quad h_- = (a_-/\alpha)h_+,$$

where $\alpha = a_+(\lambda_0)$, $\beta = b_+(\lambda_0)$ and λ_0 is the unique solution to $a_+ b_+(\lambda) = a_- b_-$.

Where does the energy required to sustain the motion come from?

[END OF PAPER]