UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 1.30 to 4.30

PAPER 73

LARGE-SCALE ATMOSPHERE-OCEAN DYNAMICS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Cartesian coordinates (x, y, z) are used with z denoting the upward vertical. The corresponding velocity components are (u, v, w).

Unless stated otherwise, g is the gravitational acceleration, f_0 is the Coriolis parameter at some latitude and β is the latitudinal gradient of the Coriolis parameter. N_0 is the buoyancy frequency, assumed to be constant unless otherwise stated.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The shallow-water system in a frame rotating about the vertical axis at rate $\frac{1}{2}f$ is governed by the equations

$$\mathbf{u}_t + \mathbf{u}.\nabla\mathbf{u} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta$$

$$\eta_t + \nabla.((\eta + H)\mathbf{u}) = 0$$

where $\mathbf{u} = (u(x,y,t), v(x,y,t), 0)$ is the velocity, $\mathbf{f} = (0,0,f)$ and the thickness of the layer is equal to $H + \eta(x,y,t)$, where H is constant.

Assume that the perturbation thickness η is small enough that the equations may be linearised about the state $\eta = 0$, $\mathbf{u} = (0,0,0)$. Show that the linearised potential vorticity, $v_x - u_y - fH^{-1}\eta$, is independent of time and, on the assumption that the initial conditions are $u = v = 0$ and $\eta = \eta_0(x)$ at $t = 0$, derive the equation governing the time evolution of η.

Making reference to this equation, briefly describe the adjustment from the initial condition to a steady state, noting the properties of any waves that propagate during the adjustment process.

Write down the equation for η in the steady state limit as $t \to \infty$ and solve it in the case

$$\begin{aligned} \eta_0(x) &= -h \qquad (x < -L) \\ \eta_0(x) &= hx/L \qquad (-L < x < L) \\ \eta_0(x) &= h \qquad (x > L) \end{aligned}$$

where h is a constant. You may find it helpful to define the parameter $\alpha \equiv Lf/(gH)^{1/2}$.

Derive the corresponding expressions for u and v in the steady state and comment on the dynamical balance. Sketch the variation of u, v and η with x in the limits $\alpha \ll 1$ and $\alpha \gg 1$ and comment on the significance of these limits. Make the magnitude of the different quantities clear in terms of h, L, f and α.

For $\alpha \ll 1$ evaluate the loss in potential energy ΔV between the initial state and the final steady state and similarly the gain in kinetic energy ΔT. (Recall that the potential energy per unit area is $g\eta^2/2H$.) Why do you expect that $\Delta T/\Delta V < 1$?

2 The Boussinesq hydrostatic form of the primitive equations on a β-plane, including the effects of a buoyancy forcing, take the form

$$u_t + (\mathbf{u}.\nabla)u - (f_0 + \beta y)v = -\frac{1}{\rho_0}\tilde{p}_x, \tag{1}$$

$$v_t + (\mathbf{u}.\nabla)v + (f_0 + \beta y)u = -\frac{1}{\rho_0}\tilde{p}_y, \tag{2}$$

$$\tilde{\rho}_t + (\mathbf{u}.\nabla)\tilde{\rho} + w\frac{d\rho_s}{dz} = r\,(x, y, z, t) \tag{3}$$

$$u_x + v_y + w_z = 0, \tag{4}$$

$$-\tilde{p}_z - g\tilde{\rho} = 0, \tag{5}$$

where the buoyancy forcing is represented by the term on the right-hand side of (3). Note that the actual density of the fluid is $\rho_0 + \rho_s(z) + \tilde{\rho}$, where ρ_0 is constant, and that $\tilde{p}$ is the pressure perturbation relative to that in a hydrostatically resting state in which the density is equal to $\rho_0 + \rho_s(z)$.

Starting from these equations, derive the corresponding form of the quasi-geostrophic potential vorticity equation. (The equation that you derive should include a term involving r.) State clearly any scaling assumptions required and approximations made. You may find it useful to assume that the flow quantities vary on a horizontal length scale L and a vertical length scale D.

A simple model of the response of the atmosphere, taken to be the half-space $z > 0$, to long-period variations in heating, is to take $r = r_0 e^{-\mu z}\cos kx \cos\omega_0 t$, where r_0, μ, k and ω_0 are all positive constants. Use the quasi-geostrophic potential vorticity equation linearised about a state of rest to analyse the response to this heating. You may apply the (artificial) boundary condition $\psi = 0$ at $z = 0$ and assume that the buoyancy frequency N is constant in height.

[Hint: you may find it useful to note that $\cos kx \cos\omega_0 t = \frac{1}{2}\mathrm{Re}\{\exp(ikx - i\omega_0 t) + \exp(ikx + i\omega_0 t)\}$ and therefore to seek solutions of the form $\psi(x, z, t) = \mathrm{Re}\{\hat{\psi}_1(z)\exp(ikx - i\omega_0 t) + \hat{\psi}_2(z)\exp(ikx + i\omega_0 t)\}$.]

In particular you should write down equations governing the vertical structure of the disturbances and solve them. Justify carefully any boundary conditions that you apply as $z \to \infty$. Comment on the difference between the cases $\omega_0 < \beta/k$ and $\omega_0 > \beta/k$.

3 Consider quasi-geostrophic Boussinesq flow on an f-plane, with constant Coriolis parameter f_0 and constant buoyancy frequency N_0. Explain, without detailed derivation of the quasi-geostrophic equations, why the leading-order approximation to the vertical velocity w, is given by

$$w = -\frac{D_g}{Dt}\left\{\frac{f_0}{N_0^2}\psi_z\right\}$$

where ψ is the quasi-geostrophic streamfunction and D_g/Dt denotes the rate of change following the geostrophic flow. Show that the appropriate boundary condition on ψ at the rigid sloping boundary $z = \alpha y$, where α is comparable to the Rossby number, is

$$\frac{D_g}{Dt}\left\{\frac{f_0\psi_z}{N_0^2}\right\} + \alpha\psi_x = 0,$$

assuming that αy is small enough that the boundary condition may be linearised and applied at $z = 0$.

A basic flow $(u, v, w) = (\Lambda z, 0, 0)$, where Λ is constant, is confined between rigid boundaries $z = \alpha y$ (linearised to $z = 0$) and $z = \gamma y + D$ (linearised to $z = D$).

Show from the above, and the quasi-geostrophic potential vorticity equation, that the equation governing the evolution of small-amplitude disturbances to this flow is

$$q'_t \equiv \left(\psi'_{xx} + \psi'_{yy} + \psi'_{zz} f_0^2/N_0^2\right)_t = 0 \quad \text{in} \quad 0 < z < D,$$

with

$$\psi'_{zt} - \Lambda\psi'_x + \frac{N_0^2\alpha}{f_0}\psi'_x = 0 \quad \text{on} \quad z = 0$$

and

$$\psi'_{zt} + \Lambda D\psi'_{zx} - \Lambda\psi'_x + \frac{N_0^2\gamma}{f_0}\psi'_x = 0 \quad \text{on} \quad z = D,$$

where disturbance quantities are denoted with primes.

By integrating $\psi'_x q'$ over the domain, assuming periodic boundary conditions in x and y, show that

$$\frac{d}{dt}\int dxdy\left\{\frac{\frac{1}{2}{\psi'}_z^2|_{z=D}}{(\Lambda f_0 - N_0^2\gamma)} - \frac{\frac{1}{2}{\psi'}_z^2|_{z=0}}{(\Lambda f_0 - N_0^2\alpha)}\right\} = 0.$$

Deduce that

$$\left(1 - \frac{N_0^2\gamma}{f_0\Lambda}\right)\left(1 - \frac{N_0^2\alpha}{f_0\Lambda}\right) > 0$$

is a necessary condition for instability.

Now consider the special case where $\gamma = \alpha$. Consider disturbances of the form $\mathrm{Re}\{\hat{\psi}(z)e^{ik(x-ct)}\}$ and show that the dispersion relation for the non-dimensionalised phase speed $\tilde{c} = c/\Lambda D$ is

$$\tilde{c} = \tfrac{1}{2} \pm \left(\tfrac{1}{4} + \tilde{\alpha}\frac{\coth\mu}{\mu} + \frac{\tilde{\alpha}^2}{\mu^2}\right)^{1/2},$$

where $\mu = N_0 kD/f_0$ and $\tilde{\alpha} + 1 = N_0^2\alpha/f_0\Lambda$. (You will almost certainly find it useful to introduce $\tilde{c}$, μ and $\tilde{\alpha}$ at an early stage in your working.)

Deduce that these disturbances do not grow exponentially in time for $\tilde{\alpha} \geq 0$ and comment briefly on the relevance or irrelevance of the condition for instability derived earlier.

4 The Boussinesq f-plane form of the Eulerian-mean equations including momentum flux and density flux as forcing terms are as follows,

$$\overline{u}_t - f_0\overline{v}_a = -(\overline{u'v'})_y \tag{1}$$

$$f_0\overline{u} = -\frac{\overline{p}_y}{\rho_0} \tag{2}$$

$$\overline{\rho}g = -\overline{p}_z \tag{3}$$

$$\overline{v}_{ay} + \overline{w}_{az} = 0 \tag{4}$$

$$\overline{\rho}_t + \overline{w}_a\frac{d\rho_s}{dz} = -(\overline{\rho'v'})_y. \tag{5}$$

Overbars in these equations indicate averages in x, primes indicate disturbance quantities, i.e. departures from the x-average value; $(\overline{v}_a, \overline{w}_a)$ are the (y, z) components of the Eulerian-mean flow, ρ is the departure of the density from the constant background value of ρ_0, and p the corresponding pressure anomaly.

Starting from these equations, derive the transformed Eulerian-mean equations. Explain the role of Eliassen-Palm flux in the transformed Eulerian-mean equations and its relation to Rossby-wave propagation. State and explain a corresponding 'non-acceleration' theorem. (Detailed derivation of the Eliassen-Palm wave-activity relation is not required.)

Consider the effect on the mean flow of propagating and dissipating Rossby waves in the domain $0 < y < L$, $-\infty < z < \infty$, with rigid walls at $y = 0$ and $y = L$. Assume the waves give rise to a buoyancy flux $\overline{\rho'v'} = \sin(\pi y/L)\mathcal{F}(z)$, where $\mathcal{F}(z) = -1$ for $z < 0$ and $\mathcal{F}(z) = 0$ for $z > 0$. Calculate and describe the resulting Eulerian-mean and transformed Eulerian-mean circulations in the (y, z) plane as represented, respectively, by streamfunctions $\mathcal{X}(y, z)$ and $\mathcal{X}^*(y, z)$. You should clearly display the equations and boundary conditions that govern each circulation, in as simple a form as possible, and sketch the corresponding solutions.