UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 1.30 to 3.30

PAPER 72

THEORY OF ELASTIC SOLIDS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Given the identity

$$\left(\frac{\partial W}{\partial A_{i\alpha}} - N_{\alpha i}\right)\dot{A}_{i\alpha} = 0$$

for any admissible $\dot{A}_{i\alpha}$, relating nominal stress $N_{\alpha i}$ and deformation gradient $A_{i\alpha}$ in elastic material with stored energy function $W(\mathbf{A})$ per unit undeformed volume, deduce that

$$N_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}} + q\frac{\partial F}{\partial A_{i\alpha}},$$

where q is an undetermined scalar, if the material is constrained so that $F(\mathbf{A}) = 0$.

If the material is incompressible, so that $\det(\mathbf{A}) \equiv (1/6)\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}A_{i\alpha}A_{j\beta}A_{k\gamma} = 1$, show that

$$N_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}} + q(\mathbf{A}^{-1})_{\alpha i}.$$

A cube, whose edges are aligned with the coordinate axes, composed of (incompressible) neo-Hookean material, with energy function

$$W(\mathbf{A}) = \frac{1}{2}\mu(A_{i\alpha}A_{i\alpha} - 3),$$

is subjected to dead-loading at its surface, to generate the uniform nominal stress $N_{11} = N_{22} = N_{33} = T$, $N_{\alpha i} = 0$ otherwise. Show that the deformation $\mathbf{A} = \mathbf{I}$ is always possible and that seven distinct homogeneous deformations are possible if $T/\mu > 3/2^{2/3}$. [*You will need to consider solutions of* $\lambda^3 - (T/\mu)\lambda^2 + 1 = 0$ *or some equivalent equation.*]

2 A right circular cylinder of radius a is composed of isotropic incompressible elastic material with stored energy function $W(\lambda_1, \lambda_2, \lambda_3)$, where λ_i are the principal stretches. The cylinder is subjected to the field of torsional deformation

$$x_1 = \xi_1\cos(\alpha\xi_3) - \xi_2\sin(\alpha\xi_3),\ x_2 = \xi_1\sin(\alpha\xi_3) + \xi_2\cos(\alpha\xi_3),\ x_3 = \xi_3. \qquad (1)$$

By considering conservation of energy or otherwise, deduce that the couple on the upper end of the cylinder required to produce the deformation (1) must have magnitude

$$M(\alpha) = 2\pi\frac{d}{d\alpha}\int_0^a W(\lambda, 1/\lambda, 1)\rho\, d\rho,$$

where

$$\lambda = \frac{\alpha\rho}{2} + \left(1 + \frac{\alpha^2\rho^2}{4}\right)^{1/2}$$

and $\rho = (\xi_1^2 + \xi_2^2)^{1/2}$. [*You are not required to prove that the given deformation is sustainable by tractions applied only over the ends of the cylinder.*]

Evaluate $M(\alpha)$ in the case of neo-Hookean material, so that $W(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$.

3 Given the equations of motion

$$N_{\alpha i,\alpha} + \rho_0 g_i = \rho_0 \ddot{x}_i \text{ in } \Omega_0$$

with boundary conditions

$$\textit{either} \quad \nu_\alpha N_{\alpha i} = t_i^0 \quad \textit{or} \quad x_i = x_i^0 \text{ on } \partial\Omega_0,$$

develop corresponding equations for an additional increment δx_i of deformation, assuming that the material is elastic with stored energy function $W(\mathbf{A})$.

(a) Show that, if the material is uniform and the initial deformation is affine and static, then the equations of motion for an increment δx_i, in the absence of body force, admit plane-wave solutions of the form

$$\delta x_i = m_i f(t - \nu_\alpha \xi_\alpha / c)$$

for arbitrary f, where

$$c_{\alpha i \beta j} \nu_\alpha \nu_\beta m_j = \rho_0 c^2 m_i,$$

in which the constants $c_{\alpha i\beta j}$ denote the moduli relating increments of nominal stress and deformation gradient.

(b) Given that the initial deformation (now not necessarily uniform) is static and maintained by time-independent dead-load tractions t_i^0 and body-forces g_i, show that a small time-harmonic perturbation of the form

$$\delta x_i = u_i(\xi_j) \exp(i\omega t)$$

is possible for certain values of ω^2. Prove that these values are real, and must be positive if the quadratic form

$$a_{i\alpha} c_{\alpha i\beta j} a_{j\beta} > 0 \text{ for all } a_{i\alpha} \neq 0.$$

[*Take* $\delta t_i^0 = \delta g_i = 0$.]

(c) Consider a similar time-harmonic perturbation for the displacement boundary-value problem, with x_i^0 time-independent. Show that the possible values of ω^2 are positive if there exist constants $K_{k\gamma}$ such that the quadratic form

$$a_{i\alpha}\{c_{\alpha i\beta j} + K_{k\gamma}\varepsilon_{\alpha\beta\gamma}\varepsilon_{ijk}\}a_{j\beta} > 0 \text{ for all } a_{i\alpha} \neq 0. \tag{2}$$

[*Take* $\delta x_i^0 = \delta g_i = 0$.]

(d) Show that, if the condition (2) is satisfied, then the material locally admits three real plane waves (positive c^2), in every direction.

4 Show that static antiplane infinitesimal deformation of isotropic elastic material with shear modulus μ can be described, in the absence of body force, by the relation

$$\sigma_{23} + i\sigma_{13} = \mu F'(z),$$

where σ_{13}, σ_{23} are the only non-zero stress components and $F(z) = \phi(x_1, x_2) + iw(x_1, x_2)$; $z = x_1 + ix_2$, w is the out-of-plane displacement and ϕ is a stress function.

The remainder of this question concerns an infinite body with a crack (or cracks) on the x_1-axis. In the absence of the crack(s), the body would experience a field of antiplane deformation with associated stress components σ_{13}^A, σ_{23}^A. *Additional* stresses σ_{13}, σ_{23} and displacement w are induced by the presence of the crack(s). Develop a Hilbert problem for the associated complex function $F'(z)$.

(a) If the crack occupies the segment $(-a, a)$ of the x_1-axis, show that when $x_1 > a$ and $x_2 = 0$,

$$\sigma_{23} + i\sigma_{13} = \frac{1}{\pi(x_1^2 - a^2)^{1/2}} \int_{-a}^{a} \frac{(a^2 - t^2)^{1/2} \sigma_{23}^A(t, 0)\, dt}{x_1 - t}.$$

(b) If cracks occupy the semi-infinite segments $(-\infty, -a)$ and (a, ∞) of the x_1-axis, show that when $-a < x_1 < a$ and $x_2 = 0$,

$$\sigma_{23} + i\sigma_{13} = \frac{1}{\pi(a^2 - x_1^2)^{1/2}} \left(\int_{a}^{\infty} - \int_{-\infty}^{-a} \right) \frac{(t^2 - a^2)^{1/2} \sigma_{23}^A(t, 0)\, dt}{t - x_1}.$$

[*The integrals are to be interpreted in the sense of Cauchy principal values. Convergence of the integrals over the semi-infinite segments may be assumed.*]