UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 1.30 to 4.30

PAPER 71

APPROXIMATION THEORY

Attempt **FOUR** *questions.*

There are **seven** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 a) For $f \in C[0,1]$, write down the definition of the Bernstein polynomial $B_n(f)$ of degree n, and prove that $\|B_n(f)\|_\infty \le \|f\|_\infty$.

b) For a function $f \in C[0,1]$ that takes integer values at $x = 0$ and $x = 1$, and for the sequence

$$B_n^*(f,x) := \sum_{k=0}^{n} \left\lfloor \binom{n}{k} f\left(\tfrac{k}{n}\right) \right\rfloor x^k (1-x)^{n-k},$$

prove that $\|B_n(f) - B_n^*(f)\|_\infty \to 0$ as $n \to \infty$. Here, $\lfloor t \rfloor$ is the largest integer not bigger than t.

c) Hence show that a function $f \in C[0,1]$ is approximable by polynomials with integral coefficients if and only if $f(0)$ and and $f(1)$ are integers.

2 1) Define a strictly convex normed linear space $\mathbb{X}$. Prove that if $\mathcal{U}$ is a subspace of such a space $\mathbb{X}$, then, for each $f \in \mathbb{X}$, there is at most one element of best approximation to f from $\mathcal{U}$.

2) The spaces $L_p(\mathbb{T})$ with the norm $\|f\|_p := \{\int_{\mathbb{T}} |f(t)|^p \, dt\}^{1/p}$ are strictly convex if $1 < p < \infty$. Let

$$f(x) = a\cos nx + b \sin nx,$$

and let $\mathcal{T}_{n-1}$ be the subspace of trigonometric polynomials of degree $\le n-1$. Prove that the best approximation t_{n-1}^* to f from $\mathcal{T}_{n-1}$ in L_p is identically zero.

[*Hint. Consider the expression* $F(x) := f(x) - t_{n-1}^*(x)$ *and using the fact that, for any* $t \in \mathbb{R}$,

$$\int_{-\pi}^{\pi} |F(x)|^p \, dx = \int_{-\pi}^{\pi} |F(x+t)|^p \, dx\,,$$

deduce that $F(x) = F(x + \frac{2\pi}{n}) = -F(x + \frac{\pi}{n})$, *hence the conclusion.*]

3 a) State the Chebyshev alternation theorem for the element of best uniform approximation to a 2π-periodic function $f \in C(\mathbb{T})$ from $\mathcal{T}_n$, the space of all trigonometric polynomials of degree $\leqslant n$.

b) Let

$$f(x) = \sum_{k=0}^{\infty} a_k \cos 5^k x, \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty.$$

Prove that, for $5^m \le n < 5^{m+1}$, the polynomial

$$t_n(x) = \sum_{k=0}^{m} a_k \cos 5^k x$$

is the best approximant to f from $\mathcal{T}_n$ and find the value of $E_n(f)$.

4 1) Let $\mathcal{S}_k(\Delta)$ be the space of splines of degree $k-1$ spanned by the B-splines $(N_j)_{j=1}^n$ on a knot sequence $\Delta = (t_j)_{j=1}^{n+k}$ such that $t_j < t_{j+k}$. Let $x = (x_i)_{i=1}^n$ be interpolation points obeying the conditions

$$N_i(x_i) > 0\,,$$

and let $P_x : C[a,b] \to \mathcal{S}_k(\Delta)$ be the map which associates with any $f \in C[a,b]$ the spline $P_x(f)$ from $\mathcal{S}_k(\Delta)$ which interpolates f at (x_i). Prove that

$$\|P_x\|_{L_\infty} \le \|A_x^{-1}\|_{\ell_\infty}$$

where A_x is the matrix $(N_j(x_i))_{i,j=1}^n$.

2) Consider the case of quadratic interpolating splines on the uniform knot-sequence $(t_1, t_2, \ldots, t_{n+3}) = (1, 2, \ldots, n+3)$ with the interpolating points

$$x_i = \tfrac{1}{2}(t_i + t_{i+3}) = i + 3/2, \quad i = 1, \ldots, n\,.$$

a) Using the recurrence relation between linear and quadratic B-splines, or otherwise, determine the values of N_j at the points (x_i).

b) Write down the matrix $A_x = (N_j(x_i))$, and evaluate the norm $\|A^{-1}\|_{\ell_\infty}$. (You may use any appropriate theorem on the inverse of certain matrices if correctly stated).

c) Hence show that $\|P_x\|_{L_\infty} \le 2$.

5 Let σ_n be the Fejer operator, i.e., for a 2π-periodic function $f \in C(\mathbb{T})$,

$$\sigma_n(f,x) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)F_n(t)\,dt, \qquad F_n(t) := \tfrac{1}{2n}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}, \qquad \tfrac{1}{\pi}\int_{-\pi}^{\pi} F_n(t)dt = 1.$$

Prove the estimate

$$\|\sigma_n(f) - f\|_\infty \le c\,\omega_2(f, \tfrac{1}{\sqrt{n}})\,,$$

where $\omega_2(f,\delta)$ is the second modulus of smoothness of f. Hence prove that if f'' is continuous, then

$$\|\sigma_n(f) - f\|_\infty = \mathcal{O}(\tfrac{1}{n})\,.$$

6 Given $k, n \in \mathbb{N}$ and a knot-sequence $\Delta = (t_i)_{i=1}^{n+k}$, let

$$Q(f, x) = \sum_{i=1}^{n} \lambda_i(f) N_i(x)$$

be the quasi-interpolant. Here λ_i is the Hahn-Banach extension of the de' Boor-Fix functional, so that

$$|\lambda_i(f)| \le c_k \, \|f\|_{C[t_i, t_{i+k}]} \quad \text{and} \quad Q(s) = s \quad \text{for all } s \in \mathcal{S}_k(\Delta)\,.$$

Prove that

$$\|Qf\|_{C[t_j, t_{j+1}]} \le c_k \|f\|_{C[t_{j+1-k}, t_{j+k}]}\,,$$

hence derive that

$$\|f - Qf\| = \mathcal{O}(|t|^k) \quad \forall\, f \in C^k[a, b],$$

where $|t| := \max |t_{i+1} - t_i|$.

7 Prove the Schoenberg-Whitney theorem: If $(t_1, \ldots, t_{n+k})$ and $(x_1, \ldots, x_n)$ are strictly increasing, then

$$A_x := \big(N_j(x_i)\big)_{i,j=1}^{n} \text{ is invertible} \quad \Leftrightarrow \quad N_i(x_i) > 0 \quad \forall i.$$

You may use the fact that, for $s = \sum_{i=p}^{p+r} N_i$ that does not vanish identically on any subinterval of $I = (t_p, t_{p+r+k})$, the number of distinct zeros of s on I is not bigger than r.