MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 9 to 12

PAPER 67

PHYSICAL COSMOLOGY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 **(a)** Assume a flat universe with a matter component and a cosmological constant Λ. By using the Friedmann equation derive the evolution of the scale factor $a(t)$ for the three cases, $\Lambda > 0$, $\Lambda < 0$ and $\Lambda = 0$ and sketch $a(t)$ for these. If $a(t)$ is bound, give the value of the extrema.

For $\Lambda > 0$ discuss the behaviour at early and late times.

(b) The luminosity distance is given by

$$d_L = \frac{r_1}{a} \ , \qquad (*)$$

where r_1 is the radial parameter distance in the Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} \ + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] ,$$

where K is the curvature.

Discuss the motivation for relation $(*)$ and show

$$d_L = \frac{1+z}{\sqrt{|\Omega_K|}H_0} S_K\left(H_0\sqrt{|\Omega_K|}\int\limits_0^z \frac{dz'}{H(z')}\right) ,$$

with $\Omega_K = -\frac{K}{H_0^2}$; H_0 the Hubble constant, $H(z) = \frac{\dot{a}}{a}$ the Hubble parameter and

$$S_k(x) = \begin{cases} \sin x & \text{if } \Omega_K < 0 \\ x & \text{if } \Omega_K = 0 \\ \sinh x & \text{if } \Omega_K > 0 \end{cases} .$$

(c) Assume a flat universe with a single component with an equation of state

$$p = w\rho \qquad (w = \text{const.}\ ;\ -1 < w < 1) \ .$$

Show that the angular diameter distance $d_A(z) = (1+z)^{-2}\ d_L(z)$ has always a turning point for $w > -1$. At which redshift z_m is this turning point? Sketch $z_m(w)$ in the region $-1 < w < 1$.

[You may take $c = \hbar = k_B = 1$ throughout.]

2 **(a)** Show that in a spherical region of a universe with only a matter component with $\Omega_m > 1$ the scale factor a reaches a maximal value of

$$a_m = \frac{\Omega_m}{\Omega_m - 1} \ ,$$

and the evolution is given by the parametric form

$$\begin{aligned} a &= (1 - \cos\theta)\,\frac{\Omega_m}{2\,(\Omega_m - 1)} \\ H_0 \cdot t &= (\theta - \sin\theta)\,\frac{\Omega_m}{2\,(\Omega_m - 1)^{3/2}} \ , \end{aligned}$$

where H_0 is the Hubble constant and t the time.

(b) The linearised scale factor a_{lin} is given by

$$\frac{a_{\text{lin}}}{a_m} = \frac{1}{4}\left(6\pi\frac{t}{t_m}\right)^{2/3}\left[1 - \frac{1}{20}\left(6\pi\frac{t}{t_m}\right)^{2/3}\right] \ ,$$

with t_m the time corresponding to the maximal expansion. What is the interpretation of the factor outside square brackets?

If we assume that the universe outside the spherical region is matter dominated and at the critical density, what is the over-density δ_{lin}, in the linear approximation in the spherical region with respect to the background density at the time of maximum expansion?

What happens to the spherical region in the further full non-linear evolution and at what time?

What is the value of the linear and non-linear, δ_{nonlin}, over-density at this point?

(c) What is the non-linear density contrast at maximum expansion (turnaround) of the spherical region?

Determine, using the virial theorem $2T + U = 0$, with T the kinetic energy and U the potential energy, the radius (in terms of the maximum radius of the region) at which the spherical region is expected to fulfil the virial theorem.

If we assume the virial theorem is fulfilled at $t = 2t_m$, what is the value of the non-linear over-density at this time?

(d) Assume that the probability that a given point in space has a linear over-density between $\delta + d\delta$, smoothed on a mass scale M is

$$p_M(\delta) = \frac{1}{\sqrt{2\pi}\sigma(M,z)}\ \exp\left\{-\frac{1}{2}\frac{\delta^2}{\sigma^2(M,z)}\right\} \ ,$$

with $\sigma(M,z)$ the variance of the density fluctuations at redshift z smoothed over a mass scale M. Further assume that clusters only form in regions with linear over-densities $\delta > \delta_c = \delta_{\text{lin}}\,(2t_m)$ and that all the matter in the universe is in clusters.

Under these circumstances show that the comoving number density of clusters of mass M at redshift z is

$$\left|\frac{dn(M,z)}{dM}\right| = \left|\sqrt{\frac{2}{\pi}}\ \frac{\rho_{m,0}}{M}\ \frac{\delta_c}{\sigma^2(M,z)}\ \frac{d\sigma(M,z)}{dM} \right.$$
$$\left. \times \exp\left(-\frac{\delta_c^2}{2\sigma^2(M,z)}\right)\right| \ ,$$

with $\rho_{m,0}$ the matter density today.

[You may take $c = \hbar = k_B = 1$ throughout.]

3 The Sunyaev-Zel'dovich (SZ) effect of a galaxy cluster is governed by the Kompaneets equation in the limit where the electron gas temperature, T_e, is much larger than the temperature of the photons, T_γ.

$$\frac{1}{\sigma_T \; n_e c} \; \frac{\partial n_\gamma}{\partial t} = \frac{\langle v^2 \rangle}{3c^2} \left[\nu^2 \frac{\partial^2 n_\gamma}{\partial \nu^2} + 4\nu \frac{\partial n_\gamma}{\partial \nu} \right] \;, \qquad (*)$$

where σ_T is the Thomson scattering cross section, n_e the electron number density, c the speed of light, n_γ the photon number density, ν the frequency and $\langle v^2 \rangle$ the rms velocity of the hot electron gas.

(a) By assuming the electron gas is described by a Maxwell-Boltzmann distribution, show that the Kompaneets equation in $(*)$ can be written as:

$$\frac{\partial n_\gamma}{\partial y} = x^{-2} \; \frac{\partial}{\partial x} \left[x^4 \; \frac{\partial n_\gamma}{\partial x} \right] \;, \qquad (**)$$

with $x = \frac{h\nu}{k_B T_\gamma}$ and ∂y defined in an appropriate way. What is the interpretation of y, and from which macroscopic gas quantity does it depend?

(b) Assume now that the undisturbed photons are given by a Planck distribution with

$$n_\gamma^0 = \frac{1}{e^{h\nu/k_B T} - 1} \;.$$

Show that small deviations from this distribution due to the Compton scattering of the photons with the electrons are given by:

$$\frac{\Delta n_\gamma}{n_\gamma^0} = \frac{e^x}{e^x - 1} x \cdot y \left\{ \frac{x}{\tanh \frac{x}{2}} - 4 \right\} \;,$$

with $y = \int^t dy$ the line of sight integral through the cluster. Show that in the Rayleigh-Jeans limit

$$\frac{\Delta n_\gamma}{n_\gamma^0} \longrightarrow -2y \;,$$

and in the Wien limit

$$\frac{\Delta n_\gamma}{n_\gamma^0} \longrightarrow x^2 y \;.$$

Discuss qualitatively the change of the spectrum. What would you expect if the cluster is moving along the line of sight?

(c) The received x-ray flux from a cluster is

$$S_x \propto \int \frac{n_e^2 \; T_e^{1/2}}{d_L(z)} \; dV \;,$$

with the integral over the volume of the cluster, $d_L(z)$ is the luminosity distance to the cluster. The SZ temperature decrement in the Rayleigh-Jeans region is

$$\Delta T_\gamma \propto \int n_e \, T_e \, d\ell \ ,$$

with the integral through the cluster along the line of sight. Assume that we obtain the redshift, z, the temperature T_e and the angular size θ by optical and x-ray (spectral) measurements. By relating $d\ell = \Delta\ell$ and $dV = \Delta V \propto \Delta\ell^3$ to the angular size of a spherical cluster, show that for low redshifts

$$H_0 \propto \tfrac{1}{\Delta T_\gamma^2} \ .$$

Discuss what could be problematic for such a measurement of H_0 with realistic clusters.

4 **(a)** Define the neutral hydrogen column density distribution of Lyman alpha absorbers.

Using the Friedmann equation for a pressureless cosmological model

$$\frac{\dot{a}^2}{a^2} + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3c^2}\rho,$$

where a is the scale factor, K is the curvature, Λ is the cosmological constant, ρ is the density, the other symbols have their usual meaning and a dot denotes differentiation with respect to the proper time, show that for a uniform comoving population of absorbers with constant cross-section, the probability that the line of sight to a distant quasar intersects such an absorber per unit redshift at redshift z is proportional to:

$$\frac{(1+z)^2}{[\Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_K(1+z)^2]^{1/2}}$$

where $\Omega_m + \Omega_\Lambda + \Omega_K = 1$ and Ω_m, Ω_Λ, and Ω_K are constants.

(You may assume $(1+z) = a_0/a$, where a_0 is the scale factor at the observer.)

(b) Give a reasonable functional form for the column density distribution at redshift $z \simeq 3$, and describe its major features with the aid of a sketch. In an Einstein-de Sitter universe, with $\Omega_m = 1$, $\Omega_\Lambda = 0$, how would you expect, on the basis of your answer to (a), the number of Lyman alpha absorbers—per unit redshift interval and with neutral hydrogen column density above a given threshold value—to change between $z = 1.5$ and $z = 4$? Briefly discuss how this compares with what is observed.

(c) Explain, using appropriate expressions, how the column density distribution can be used to infer the mass density of (i) neutral gas and (ii) total (neutral+ionised) gas at $z = 3$, both expressed as a fraction of the critical density $\rho_{\rm crit}$.

(d) Comment on the values found, comparing (ii) with the total density of baryons and (i) with the density of luminous matter today. Briefly discuss the limitations of this kind of calculation.