UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 1.30 to 4.30

PAPER 65

DYNAMO THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A model of a mean-field dynamo in the presence of shear is given by the equations

$$\mathbf{B}_t + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \alpha \nabla \times \mathbf{B} + \eta \nabla^2 \mathbf{B},$$

where α is a constant and where the simple shear flow $\mathbf{u} = T^{-1}(y, 0, 0)$. Verify that this equation has a solution of the form

$$\mathbf{B} = Re\left[(R(t)\hat{\mathbf{z}} \times \mathbf{k}(t) + S(t)\hat{\mathbf{z}}) e^{i\mathbf{k}(t)\cdot\mathbf{x}} \right],$$

where $\mathbf{k}(t) = k_0(1, -t/T, 0)$, and R, S obey the equations

$$R_t = -i\alpha S - \eta k^2 R, \quad S_t = i\alpha k^2 R - \eta k^2 S, \quad k^2 = |\mathbf{k}|^2.$$

Given that for $t \gg T$ an approximate solution to the pair of equations $G_t = -iH,\ H_t = ik^2 G$ is $G \sim k^{-\frac{1}{2}} e^{\int k dt},\ H \sim kG$, show that when η is sufficiently small (in a sense to be determined) the magnetic energy at first increases and then decays at large times.

2 A model for an "$\alpha^2\Omega$" dynamo, analogous to the Parker wave model, is given by the equations (where $A = A(x,t), B = B(x,t)$):

$$\begin{aligned} A_t &= \alpha B + \eta(A'' - \ell^2 A) \\ B_t &= \Omega A' - \alpha(A'' - \ell^2 A) + \eta(B'' - \ell^2 B) \end{aligned}$$

where ℓ,α,Ω are constants, and the primes denote derivatives w.r.t. x.

(i) Explain how the terms in these equations may be derived from the full mean field induction equation by appropriate simplifications.

(ii) Give an equation for the (complex) growth rate $\sigma = \sigma(\alpha, \Omega, k, \ell)$ of solutions with $A, B \propto e^{ikx}$, and for fixed ℓ, Ω find the value k_c of k giving the smallest value of $|\alpha|$ for marginally stable solutions (Re $\sigma = 0$). Note that $k_c = 0$ when $\Omega^2 < \Omega_0^2$, where Ω_0^2 is to be determined. Sketch the graphs of $|\alpha|$ versus k in the two cases $\Omega^2 > \Omega_0^2$, $\Omega^2 < \Omega_0^2$.

3 Consider the evolution of a two-dimensional magnetic field $\mathbf{B} = B(x,y,t)\hat{\mathbf{z}} + \nabla \times A(x,y,t)\hat{\mathbf{z}}$ in the presence of a solenoidal velocity field $\mathbf{u}(x,y)$, in a fluid of constant diffusivity η. All fields are considered periodic with period $2\pi/k$ in x and y with zero mean. It may be assumed that $\langle|\nabla A|^2\rangle \geqslant k^2\langle A^2\rangle$, where the brackets indicate averages over the periodic domain, and similarly for $\langle B^2\rangle$.

(i) Show that if $\langle A^2\rangle = A_0^2$ at $t = 0$, then $\langle A^2\rangle \leqslant A_0^2 e^{-2\eta k^2 t}$.

(ii) Show by using the Schwarz inequality that

$$\frac{d}{dt}\frac{1}{2}\langle B^2\rangle \leqslant Q\sqrt{\langle A^2\rangle\langle|\nabla B|^2\rangle} - \eta\langle|\nabla B|^2\rangle,$$

where Q is the maximum value of $|\nabla(\mathbf{u}\cdot\hat{\mathbf{z}})|$. Deduce by maximizing the right hand side of this expression as a function of $\langle|\nabla B|^2\rangle$ that $\langle B^2\rangle$ can increase by no more than $Q^2A_0^2/4k^2\eta$.

(iii) Show further that $\langle B^2\rangle$ ultimately decays to zero.

4 Write an essay on Taylor's condition, and its use in solving the magnetostrophic equation. Include discussions of the cases of zero and small viscosity.