UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 1.30 to 4.30

PAPER 63

STRUCTURE AND EVOLUTION OF STARS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

The notation used is standard and you are reminded of the equations of stellar structure in the form:

$$\frac{dP}{dr} = -\frac{\mathrm{G}m\rho}{r^2};$$

$$\frac{dm}{dr} = 4\pi r^2 \rho;$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi a c r^2 T^3};$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho\epsilon;$$

$$P = \frac{\Re\rho T}{\mu} + \frac{1}{3}aT^4.$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A zero-age main-sequence model of the Sun is fully radiative. Its material behaves as an ideal gas and has opacity

$$\kappa = \kappa_0 Z \rho T^{-3},$$

where κ_0 is a constant, ρ and T are its density and temperature and Z is the mass fraction of metals. The energy generation rate per unit mass is

$$\epsilon = \epsilon_0 X^2 \rho T^5,$$

where ϵ_0 is a constant and X is the mass fraction of hydrogen. Show that, for small Z, the mean molecular weight

$$\mu \approx \frac{4}{5X+3}$$

and that the luminosity L and central temperature $T_{\rm c}$ obey

$$L \propto \frac{\mu^7}{Z} \quad \text{and} \quad T_{\rm c} \propto \frac{\mu^{5/4}}{X^{1/4} Z^{1/8}}.$$

Two such models of the Sun have the same luminosity $L = L_\odot$ but differ in composition. The first has $X = X_1 = 0.7$ and $Z = Z_1 = 0.02$ while the second has $X = X_2$ and $Z = Z_2 = 0.01$. Find X_2 to one significant figure and determine which model has the higher central temperature.

The energy released in burning a unit mass of hydrogen to helium is E_0. Assuming that the stars remain homogeneous, show that the luminosity varies with time t as

$$L(t) = L_0 \left(1 - \frac{10\mu_0 L_0 t}{E_0 M_\odot}\right)^{-\frac{7}{8}},$$

where L_0 and μ_0 are the luminosity and mean molecular weight at $t = 0$.

[You may find it useful to know that $2^{1/7} \approx 1.1$*]*

2 In a plane-parallel grey atmosphere of negligible mass and containing no sources of energy the optical depth τ is defined by $d\tau = -\kappa\rho\, dz$, where $\kappa(\rho, T)$ is the total opacity of stellar material of density ρ and at temperature T, z is the height in the atmosphere and $\tau \to 0$ at large z. The equation of radiative transfer can be written in the form

$$\cos\theta \frac{dI}{d\tau} = I - \frac{j}{\kappa}, \qquad (*)$$

where $I(\tau, \theta)$ is the intensity of radiation in at optical depth τ at an angle θ to the z-axis and j, the effective emissivity including scattering and spontaneous emission is isotropic and so given by

$$\frac{j}{\kappa} = \frac{\sigma T^4}{\pi},$$

where σ is the Stefan–Boltzmann constant. Integrate $(*)$ over a sphere and use the fact that the flux F in the z direction is independent of τ to deduce that

$$4\pi \frac{j}{\kappa} = \int_{\text{sphere}} I(\tau, \theta)\, d\Omega = 4\pi J,$$

where $J(\tau)$ is the mean intensity.

Show that the form

$$I(\tau, \theta) = A(\tau) + C(\tau)\cos\theta$$

satisfies the Eddington closure approximation

$$cP_{\mathrm{r}} = \frac{4}{3}\pi J$$

between radiation pressure $P_{\mathrm{r}}(\tau)$, the speed of light c and the mean intensity, and is a solution to $(*)$ if

$$\frac{dA}{d\tau} = C$$

and that

$$C = \frac{3F}{4\pi}.$$

Use the fact that there is no flux into the star at $\tau = 0$,

$$F_{\text{in}} = \int_{\text{inwardhemisphere}} I \cos\theta\, d\Omega = 0,$$

to find $A(\tau)$ and use the definition $F = \sigma T_{\mathrm{e}}^4$ of effective temperature T_{e} to deduce that

$$T^4 = \frac{3}{4} T_{\mathrm{e}}^4 \left(\tau + \frac{2}{3}\right),$$

and that when $\tau = 0$, $T = T_0 = 2^{-1/4} T_{\mathrm{e}}$.

In the atmosphere of a red dwarf the opacity obeys

$$\kappa = \kappa_0 P^{\alpha-1} T^{4-4\beta}$$

and radiation pressure is negligible. By considering hydrostatic equilibrium show that the pressure P varies with temperature as

$$P^{\alpha} = \frac{2\alpha g}{3\kappa_0 T_0^4}(T^{4\beta} - T_0^{4\beta}),$$

where g is the surface gravity of the star.

Hence deduce that an appropriate surface boundary condition, for the stellar interior, is

$$\frac{P\kappa}{g} = \frac{4\alpha}{3\beta}(1 - 2^{-\beta})$$

at the location where $L_r = 4\pi\sigma r^2 T^4$, where L_r is the luminosity at radius r from the centre of the star.

3 A cataclysmic variable consists of a white dwarf of mass M_1 and a low-mass main-sequence companion of mass M_2 in a circular orbit with separation a. The main-sequence star is filling its Roche lobe and transferring mass to the white dwarf at a rate $\dot{M}_1 \approx -\dot{M}_2$. The mass ratio $q = M_2/M_1 < 1$. The hydrostatic and thermal equilibrium radius of the main-sequence star can be approximated by

$$\frac{R_2}{R_\odot} = \frac{M_2}{M_\odot},$$

while for a suitable range of mass ratios the Roche-lobe radius R_L obeys

$$\frac{R_\mathrm{L}}{a} = 0.46 \left(\frac{M_2}{M}\right)^{\frac{1}{3}},$$

where $M = M_1 + M_2$. Show that the period P of the binary is given by

$$\frac{P}{P_0} = \frac{M_2}{M_\odot}$$

for some constant P_0.

The spin angular momentum of the stars can be neglected. Show that the orbital angular momentum is

$$J = \frac{M_1 M_2}{M} a^2 \Omega,$$

where $\Omega = 2\pi/P$ is the orbital angular velocity.

Find $\dot{R}_\mathrm{L}/R_\mathrm{L}$ as a function of $\dot{M}_2/M_2$ when $\dot{J} = 0$ and compare this with $\dot{R}_2/R_2$. What would be the equilibrium response of the system to mass transfer if $q < 4/3$.

Describe briefly one mechanism that can lead to angular momentum loss ($\dot{J} < 0$) and maintain mass transfer if $q < 4/3$.

Once a layer of hydrogen-rich material of mass $\delta m \approx 10^{-4}\, M_\odot$ has accumulated on the surface of the white dwarf thermonuclear reactions ignite in the degenerate material. These expel the entire layer of mass δm from the system in a nova explosion lasting a few hundred orbital periods. Comment on the effect of this on eccentricity and show that the change in separation $\delta a/a = \delta m/M$ and the change in Roche-lobe radius $\delta R_\mathrm{L}/R_\mathrm{L} = 4\,\delta m/3M$ to first order in $\delta m/M$.

Mass transfer is interrupted and the main-sequence star responds by shrinking inside its Roche lobe. Assuming that the rate of angular momentum loss $-\dot{J}$ remains constant until the next nova explosion, show, again to first order in $\delta m/M$, that the ratio of the time spent detached t_d to the time spent semi-detached t_s is

$$\frac{t_\mathrm{d}}{t_\mathrm{s}} = \frac{2q}{(4-3q)(1+q)}.$$

4 Describe the evolution of a $5\,M_\odot$ star from the zero-age main sequence to the onset of thermal pulses. Pay particular attention to the various energy generation mechanisms and indicate timescales. Include an evolutionary track in a Hertzsprung–Russell diagram.

Without discussing the details of the thermally pulsing asymptotic branch, describe the final stages of the evolution of such a star and indicate how these might differ if the rate of mass loss were much lower than, or much higher than, expected.