UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 9 to 12

PAPER 62

STELLAR MAGNETOHYDRODYNAMICS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Candidates may bring their own notebooks into the examination.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A two-dimensional magnetic field is represented by the flux function $A(s, \phi, t)$, referred to plane polar co-ordinates (s, ϕ). State the equation governing the evolution of A in a medium with diffusivity η, in the presence of a two-dimensional velocity $\mathbf{u}$. Consider a differentially rotating flow with $\mathbf{u}_Q = s(\Omega_0 + \omega s^2)$ and an initial field given by

$$A(s, \phi, 0) = C s^m e^{im\phi} ,$$

where the integer $m \geqslant 1$, and C, Ω_0 and ω are constants. Let

$$A(s, \phi, t) = C a(s, t) e^{im\phi} ,$$

with

$$a(s, t) = s^m g(t) \exp -i[m\Omega_0 t + s^2 f(t)] .$$

Show that $f = (m\omega/\mu)\tanh \mu t$ and $g = C'(\cosh \mu t)^{-(m+1)}$, where

$$\mu = (1 + i)(2\eta m\omega)^{1/2}$$

and C' is a constant. Hence show that for $\mu t >> 1$ the field decays as $\exp(-t/\tau)$, where

$$\tau = [(m + 1)(2\eta m\omega)^{1/2}]^{-1} \propto Rm^{1/2}/\Omega_0$$

and the magnetic Reynolds number $Rm = \Omega_0^2/\eta\omega$.

2 Consider a *pressure-balancing* magnetic field satisfying

$$(\nabla \wedge \mathbf{B}) \wedge \mathbf{B} = \mu_0 \nabla p$$

where $\mathbf{B} = \mathbf{B}(r, \theta)$, $p = p(r, \theta)$ in spherical polar co-ordinates (r, θ, ϕ). Writing

$$\mathbf{B} = \Big(0, 0, \frac{\psi}{r \sin\theta}\Big) + \nabla \wedge \Big(0, 0, \frac{\chi}{r \sin\theta}\Big),$$

show that $p = p(\chi), \psi = \psi(\chi)$ and

$$\frac{\partial^2 \chi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\theta} \frac{\partial\chi}{\partial\theta}\Big) + \psi \frac{d\psi}{d\chi} = r^2 \sin^2\theta \frac{dp}{d\chi} .$$

Show that these equations have a solution for which $p \propto \chi$, $\psi \propto \chi$ and χ takes the form

$$\chi = \Big(A \cos\alpha r + \frac{B \sin\alpha r}{r} + cr^2\Big) \sin^2\theta .$$

Find conditions on A, B, C and α for which $\mathbf{B} = 0$ on the sphere $r = a$, and give an expression for $p(a, \theta)$.

3 Two-dimensional Boussinesq magnetoconvection in the presence of a *vertical* magnetic field is described by a streamfunction $\psi(x,z,t)$, a temperature fluctuation $\theta(x,z,t)$ and a flux function $\chi(x,z,t)$ that satisfy the dimensionless equations

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} &= \sigma\Big[\nabla^4\psi + R\frac{\partial\theta}{\partial x} + \zeta Q\frac{\partial(x+\chi,\nabla^2\chi)}{\partial(x,z)}\Big],\\
\frac{\partial\theta}{\partial t} + \frac{\partial(\psi,\theta)}{\partial(x,z)} &= \frac{\partial\psi}{\partial x} + \nabla^2\theta,\\
\frac{\partial\chi}{\partial t} + \frac{\partial(\psi,\chi)}{\partial(x,z)} &= \frac{\partial\psi}{\partial z} + \zeta\nabla^2\chi,
\end{aligned}
$$

in the domain $\{0 \leqslant x \leqslant \pi/k; 0 \leqslant z \leqslant 1\}$, subject to suitable boundary conditions, where R, Q, σ and ζ are the Rayleigh number, Chandrasekhar number, Prandtl number and the ratio of magnetic to thermal diffusivity, respectively.

Assuming that ψ, θ, χ can be represented by the truncated expressions

$$
\begin{aligned}
\psi &= a(\tau)\frac{(8p)^{1/2}}{k}\sin kx \sin \pi z,\\
\theta &= b(\tau)\Big(\frac{8}{p}\Big)^{1/2}\cos kx \sin \pi z - \frac{1}{\pi}c(\tau)\sin 2\pi z,\\
\chi &= \frac{1}{k}\Big[\pi d(\tau)\Big(\frac{8}{p}\Big)^{1/2}\sin kx \cos \pi z + e(\tau)\sin 2kx\Big],
\end{aligned}
$$

where $p = \pi^2 + k^2$ and $\tau = pt$, show that a, b, c, d, e satisfy the ordinary differential equations

$$
\begin{aligned}
\dot a &= \sigma[-a + rb + \zeta q d\{(\varphi-3)e - 1\}],\\
\dot b &= -b + a(1-c),\\
\dot c &= \varphi(-c + ab),\\
\dot d &= -\zeta d + a(1-e),\\
\dot e &= -(4-\varphi)\zeta e + \varphi a d,
\end{aligned}
$$

where $r = k^2R/p^3, q = \pi^2 Q/p^2$ and $\varphi = 4\pi^2/p$. Hence show that there is a stationary bifurcation from the trivial solution at $r = 1 + q$, leading to a branch of steady solutions with

$$
r = 1 + a^2 + \mu(1+a^2)(\mu+a^2)^{-2}[\mu + (4-\varphi)a^2]q,
$$

where $\mu = (4-\varphi)\zeta^2/\varphi$, and that this branch bifurcates supercritically or subcritically, depending on whether

$$
\varphi q(2-\varphi) + \zeta^2(1+q)(4-\varphi)
$$

is positive or negative.

4 Discuss the evolution of magnetic activity and rotation in a star like Sun during its lifetime on the main sequence, explaining how the stellar dynamo and the angular velocity interact with each other.