UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 7 June, 2004 1.30 to 3.30

# PAPER 61

# BOUNDARY VALUE PROBLEMS FOR INTEGRABLE PDE's

*Attempt* **TWO** *questions.*

*There are* **three** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $Q(X,T)$ satisfy the following initial-boundary value problem:

$$Q_T + Q_{XXX} + aQ_{XX} + bQ_X + cQ = 0, \qquad 0 < X < \infty, \qquad T > 0,$$

$$Q(X,0) = Q_0(X), \qquad 0 < X < \infty$$

$$Q(0,T) = G_0(T), \qquad T > 0,$$

where $a$, $b$, $c$ are real constants, $a < 0$, $b < 0$, $a^2 \neq 3b$, the function $Q_0(X)$ decays for large $X$, and the functions $Q_0$ and $G_0$ have sufficient smoothness and are compatible at $X = T = 0$, i.e. $Q_0(0) = G_0(0)$.

(a) Show that by a suitable change of variables the above problem can be reduced to the following problem:

$$q_t + q_{xxx} - q_x = 0, \qquad 0 < x < \infty, \qquad t > 0, \tag{1.1}$$

$$q(x,0) = q_0(x), \qquad 0 < x < \infty,$$

$$q(0,t) = g_0(t), \qquad t > 0,$$

and express $q_0$ and $g_0$ in terms of $Q_0$ and $G_0$.

(b) Write equation (1.1) in the form

$$(e^{-ikx+w(k)t}q)_t + (e^{-ikx+w(k)t}X)_x = 0, \qquad k \in \mathbb{C}$$

where $w(k)$, $X(x,t,k)$ are to be determined.

(c) Use the result of (b) and the Fourier transform to construct an integral representation for $q(x,t)$ in the complex $k$-plane involving appropriate spectral functions.

(d) Use the global relation to express the spectral functions in terms of the Fourier transform of $q_0(x)$ and of a $t$-transform of $g_0(t)$.

(e) Rewrite the spectral functions in a form suitable for analysing the long time behaviour of the solution.

(f) Comment briefly on the case of $a > 0$.

**2** Let $D$ be the equilateral triangle with corners at the points

$$z_1 = \frac{\ell e^{\frac{i\pi}{3}}}{\sqrt{3}}, \quad z_2 = \overline{z_1}, \quad z_3 = -\frac{l}{\sqrt{3}},$$

where $z$ denotes the usual complex variable $z = x + iy$ and $l$ is a positive constant. The sides $(z_2, z_1)$, $(z_3, z_2)$, $(z_1, z_3)$ will be referred to as sides $(1), (2), (3)$ respectively. The function $q^{(j)}(s)$ denotes $q$ on the side $(j)$ and the function $q_N^{(j)}(s)$ denotes the Neumann boundary value on the side $(j)$. Let the real-valued function $q(x,y)$ satisfy the PDE

$$q_{xx} + q_{yy} - 4\lambda q = 0, \quad (x,y) \in D \tag{2.1}$$

where $\lambda$ is a real constant, with the Dirichlet boundary conditions

$$q^{(j)}(s) = f(s), \quad s \in [-\frac{l}{2}, \frac{l}{2}], \quad j = 1,2,3 \tag{2.2}$$

where the function $f(s)$ is sufficiently smooth and satisfies the continuity condition $f(-\frac{l}{2}) = f(\frac{l}{2})$.

(a) Show that the 1-form

$$W = e^{-ikz - \frac{\lambda}{ik}\bar{z}}(q_z dz - \frac{\lambda}{ik} q d\bar{z}), \quad k \in \mathbb{C} - \{0\},$$

is closed.

(b) Show that the global relation associated with equation (2.1) is

$$E(-ik)\Psi_1(k) + E(-i\bar{a}k)\Psi_2(\bar{a}k) + E(-iak)\Psi_3(ak) =$$

$$2i[E(-ik)\Phi_1(k) + E(-i\bar{a}k)\Phi_2(\bar{a}k) + E(-iak)\Phi_3(ak)], \quad k \in \mathbb{C} - \{0\}$$

where

$$E(k) = e^{(k+\frac{\lambda}{k})\frac{l}{2\sqrt{3}}}, \quad a = e^{\frac{2i\pi}{3}},$$

$$\Psi_j(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{(k+\frac{\lambda}{k})s} q_N^{(j)}(s) ds,$$

$$\Phi_j(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{(k+\frac{\lambda}{k})s} [\frac{1}{2}\frac{d}{ds} q^{(j)}(s) + \frac{\lambda}{k} q^{(j)}(s)] ds.$$

and $q_N^{(j)}(s)$ denotes the Neumann boundary value on the side $(j)$.

(c) Show that in the particular case of the boundary conditions (2.2) the global relation is

$$e(\bar{a}k)\Psi(k) + e(-k)\Psi(\bar{a}k) + \Psi(ak) = 2iA(k), \tag{2.3}$$

and compute $e(k)$ and $A(k)$. For this derivation recall that

$$1 + a + \bar{a} = 0, \quad i\bar{a} - ia = \sqrt{3}, \quad ia - i = \sqrt{3}\bar{a}.$$

(d) Use equation (2.3) as well as the equation obtained from equation (2.3) by Schwarz conjugation to compute $q_N(s)$ in terms of $f(s)$.

**3** It can be shown that the defocusing NLS equation

$$iq_t + q_{xx} - 2|q|^2 q = 0, \tag{3.1}$$

admits the Lax pair

$$\mu_x + ik\hat{\sigma}_3\mu = Q\mu,$$
$$\mu_t + 2ik^2\hat{\sigma}_3\mu = (2kQ - iQ_x\sigma_3 - i|q|^2\sigma_3)\mu$$

where $\mu(x,t,k)$ is a $2\times 2$ matrix-valued function and $\hat{\sigma}_3$, $\sigma_3$, $Q$ are defined as follows:

$$\hat{\sigma}\mu = [\sigma_3, \mu], \sigma = diag(1,-1),$$

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{pmatrix}.$$

Let $q$ be a complex-valued function decaying as $x \to \infty$, which satisfies the defocusing NLS equation in $0 < x < \infty$, $0 < t < T$, where $T$ is a positive constant.

(a) Let $\mu_1$, $\mu_2$, $\mu_3$, be solutions of the above Lax pair normalized at $(0,T)$, $(0,0)$, $(\infty,T)$ respectively. Find the domains of the complex $k$-plane where the column vectors of these matrices are bounded and analytic.

(b) Show that the functions $\mu_j$, $j = 1,2,3$, are related by the equations

$$\mu_3(x,t,k) = \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{\sigma}}s(k),$$

$$\mu_1(x,t,k) = \mu_2(x,t,k)e^{-i(kx+2k^2t)\hat{\sigma}}S(k),$$

and express $s(k)$ and $S(k)$ in terms of $\mu_3(x,0,k)$ and $\mu_2(0,t,k)$.

(c) Discuss how the equations obtained in (b) can be used to obtain a Riemann-Hilbert problem. What is the relevant contour for this Riemann-Hilbert problem?

(d) Show that there exists a simple relation between $s(k)$ and $S(k)$.

(e) By analysing the linear limit of the relation obtained in (d), determine the number of the boundary conditions needed at $x = 0$ for the problem to be well posed, at least for a small suitable norm of $q$.

(f) Consider the following homogeneous Robin problem for eq (3.1) on the quarter plane $0 < x < \infty$, $\quad 0 < t < \infty$:

$$q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}^+),$$
$$q_x(0,t) - cq(0,t) = 0,$$

where $c$ is a real constant and $\mathcal{S}$ denotes the space of Schwartz functions.

Introduce the notations

$$a(k) = (s(k))_{22}, \qquad b(k) = (s(k))_{12}, \qquad A(k) = (S(k))_{22}, \qquad B(k) = (S(k))_{12}.$$

Find an expression for $B(k)/A(k)$ in terms of $a(k)$ and $b(k)$ and hence deduce that the Riemann-Hilbert problem discussed in (c) above is uniquely specified in terms of $q_0(x)$ and $c$.