UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 9 to 12

PAPER 60

LOCAL AND GLOBAL BIFURCATIONS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $R_{\mathcal{G}}$ be a real representation of a finite group $\mathcal{G}$ acting on $\mathbb{R}^n$.

(a) Define the terms *isotropy subgroup* and *fixed point subspace.*

(b) State the Equivariant Branching Lemma as it applies to a set of $R_{\mathcal{G}}$-equivariant nonlinear ODEs $\dot{x} = f(x, \mu)$.

(c) Consider the group Γ of rotations and reflections of a cube in $\mathbb{R}^3$, centred at the origin and aligned with the co-ordinate axes. A representation of Γ on $\mathbb{R}^3$ is defined by the following matrices representing elements that generate the group:

$$\kappa_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad r_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad r_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Show that this representation of Γ is absolutely irreducible.

(d) By determining their isotropy subgroups show that three (group orbits of) distinct equilibria are guaranteed to bifurcate from the origin in a generic bifurcation problem that is equivariant under this representation of Γ.

Hint: to simplify notation, define κ_y, κ_z and r_z by analogy with (1).

(e) Determine the normal form for the bifurcation, including terms up to cubic order.

2 (a) Consider the ODE

$$\dot{x} = \mu^2 - x^2 - \frac{1}{3}x^3. \tag{2}$$

(i) Locate three nonhyperbolic equilibrium points of (2) and sketch the location of the equilibria in the (μ, x) plane, indicating stabilities. Is there a transcritical bifurcation at $\mu = 0$? Justify your answer carefully.

(ii) Sketch bifurcation diagrams in the (μ, x) plane for the unfolded system

$$\dot{x} = \mu^2 - x^2 - \frac{1}{3}x^3 + \varepsilon, \tag{3}$$

in the two cases $\varepsilon > 0$ and $\varepsilon < 0$.

(b) Now consider the second-order system

$$\ddot{v} + \lambda v - \kappa \dot{v} = v^3 - v^2 \dot{v} + 2\varepsilon v^2,$$

where λ and κ are bifurcation parameters, and $|\varepsilon| \ll 1$. By considering the nature of the local and global bifurcations in the unperturbed case $\varepsilon = 0$, discuss the qualitative changes (for both local and global bifurcations) in the bifurcation structure that occur when $\varepsilon \neq 0$.

Sketch the sequence of bifurcations that occur along the lines $\kappa + \lambda = \pm 1$ when $\varepsilon \neq 0$.

3 In an imposed vertical magnetic field, two-dimensional Boussinesq convection in a liquid metal confined to the region $0 \leqslant x \leqslant L$, $0 \leqslant z \leqslant 1$ is governed by the reduced nondimensionalised PDEs

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x,z)} = \sigma \nabla^4 \psi + \sigma R \frac{\partial \theta}{\partial x} - \sigma Q \frac{\partial^2 \psi}{\partial z^2}, \tag{4}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial(\psi, \theta)}{\partial(x,z)} = \nabla^2 \theta + \frac{\partial \psi}{\partial x}, \tag{5}$$

where $\partial(f,g)/\partial(x,z) = \partial_x f \partial_z g - \partial_z f \partial_x g$ is the usual Jacobian expression, $\psi(x,z,t)$ is the streamfunction, $\theta(x,z,t)$ is the deviation from the linear temperature profile $T = 1-z$, σ is the Prandtl number, R is the Rayleigh number (proportional to the imposed temperature difference across the layer) and $Q > 0$ is the Chandrasekhar number (proportional to the square of the imposed magnetic field). Note that the magnetic field itself does not play a dynamical role, and hence there is no separate evolution equation for the flux function $A(x,z,t)$. The boundary conditions are that the velocity field is stress-free on the domain boundary, no heat flux is allowed across the side walls, and the temperature is fixed at the top and bottom; i.e. $\psi = \nabla^2 \psi = 0$ on the domain boundary, $\partial \theta / \partial x = 0$ on $x = 0, L$ and $\theta = 0$ on $z = 0, 1$.

(a) Either by adopting a suitable Fourier truncation, or by modified perturbation theory, propose expressions for the spatial structure of the fluid motion near the onset of convection. Hence find the critical Rayleigh number $R = R_c(Q, \alpha)$ for the onset of steady convection, where $\alpha = \pi/L$. Determine whether an initial bifurcation to oscillatory convection is possible.

(b) Show that in the asymptotic limit of large Q the critical (most unstable) wavenumber for the onset of steady convection scales as

$$\alpha \sim \left(\frac{\pi^4}{2}\right)^{1/6} Q^{1/6}.$$

(c) Show that the onset of steady convection is via a supercritical pitchfork bifurcation for all values of Q.

Explain why the results of the Fourier truncation approach and modified perturbation theory are identical.

4 The following map describes the behaviour of periodic orbits near a gluing bifurcation in a second-order set of ODEs:

$$\begin{aligned} x_{n+1} &= -\mu \operatorname{sgn}(x_n) + A \operatorname{sgn}(y_n)|x_n|^{\delta}, \\ y_{n+1} &= \operatorname{sgn}(x_n), \end{aligned} \tag{6}$$

where the gluing bifurcation occurs at $\mu = 0$, $A > 0$ and $\delta > 0$ are parameters, and $\operatorname{sgn}(x) = x/|x|$ when $x \neq 0$.

(a) Define the new variable $z_n = -x_n y_n$ and show that z_n satisfies the one-dimensional map

$$z_{n+1} = f(z_n) \;=\; \mu + A \operatorname{sgn}(z_n)|z_n|^{\delta}. \tag{7}$$

(b) Show that some fixed points of (7) correspond to fixed points of (6) while others correspond to period-two points of (6). Hence describe the behaviour of periodic orbits in the ODEs at the gluing bifurcation, in the two cases $\delta > 1$ and $\delta < 1$.

(c) Now assume that $A < 1$. Consider the case $\delta = 1 + \varepsilon$, with $|\varepsilon| \ll 1$. Use the condition for the existence of a saddle-node bifurcation in the map (7) to show that there is a cusp-shaped region of the (μ, δ) plane in which there is more than one periodic orbit. Sketch the form of the map (7) at points on the boundary of this region.

(d) Sketch bifurcation diagrams in the (μ, z) plane in the cases $\varepsilon > 0$ and $\varepsilon < 0$.