UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 1.30 to 4.30

PAPER 6

INTRODUCTION TO FUNCTIONAL ANALYSIS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that all continuous functions of two variables may be expressed as the composition of functions of one variable together with addition.

2 (i) State and prove Tychonov's theorem using Zorn's lemma and the axiom of choice. State and prove Alaoglu's theorem on the compactness of a certain unit ball.

(ii) In this part of the question we do not assume the axiom of choice. We write $\mathbb{N}$ for the positive integers. Show that the statement 'if (X_j, τ_j) $[j \in \mathbb{N}]$ are compact topological spaces, then $\prod_{j=0}^{\infty} X_j$ with the product topology is compact' implies the statement 'if the sets A_j are non-empty $[j \in \mathbb{N}]$ then there exists a function $f : \mathbb{N} \to \bigcup_{j=0}^{\infty} A_j$ with $f(j) \in A_j$'.

3 (i) Suppose that B is an algebra over $\mathbb{C}$ with a unit e. Suppose that, as a vector space, B has a complete norm $\| \ \|$. If multiplication is left and right continuous, show that there is an equivalent norm $\| \ \|_*$ such that $\|x\|_* \|y\|_* \geqslant \|xy\|_*$ for all x, $y \in B$.

(ii) Starting from the definition of a Banach algebra, show that any Banach algebra which is also a field is isomorphic as a Banach algebra to $\mathbb{C}$.

(iii) Does there exist a commutative Banach algebra B with unit and an $x \in B$ such that $x^n \neq 0$ for all $n \geqslant 1$ but the spectral radius $\rho(x) = 0$? Does there exist a commutative Banach algebra B with unit e such that, writing

$$Y = B \setminus \{\lambda e \,:\, \lambda \in \mathbb{C}\}\,,$$

we have $Y \neq \emptyset$ and such that, whenever $y \in Y$, $y^n \neq 0$ for all $n \geqslant 1$ but $\rho(y) = 0$? Give reasons.

4 (i) Show that, if $(V, \| \ \|)$ is a real normed space and E is a convex subset of V containing the open ball $B(\mathbf{0}, \epsilon)$ for some $\epsilon > 0$, then, given any $\mathbf{x} \notin E$, we can find a continuous linear map $T : V \to \mathbb{R}$ such that $T\mathbf{x} \geqslant 1 \geqslant T\mathbf{e}$ for all $\mathbf{e} \in E$. [If you use any other form of the Hahn–Banach theorem you should prove it.]

(ii) By using the result of (i), or otherwise, show that, if $(V, \| \ \|)$ is a real normed space and F is a convex subset of V with $B(\mathbf{0}, \epsilon) \cap F = \emptyset$ for some $\epsilon > 0$, then we can find a continuous linear map $S : V \to \mathbb{R}$ such that $S\mathbf{f} \geqslant 1$ for all $\mathbf{f} \in F$.

(iii) Suppose that $(V, \| \ \|)$ is a real normed space and $\mathbf{f}_n$ is sequence of points in V such that $T\mathbf{f}_n \to 0$ as $n \to \infty$ for every continuous $T : V \to \mathbb{R}$. Show, by using the result of (ii), that given any $\epsilon > 0$ we can find $N \geqslant 1$ and $\lambda_j \geqslant 0$ with $\sum_{j=1}^{N} \lambda_j = 1$ and

$$\left\| \sum_{j=1}^{N} \lambda_j \mathbf{f}_j \right\| < \epsilon\,.$$